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Convergence analysis of P_1 finite element method for free boundary problems on non-overlapping subdomains ☆

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Abstract

The paper applies a conforming P_1 finite element method to general variational inequalities derived from free boundary problems. The domain of the free boundary problem can be properly split into two non-overlapping subdomains where the free boundary is located in only one subdomain. The variational inequality is reduced to a partial differential equation in the subdomain which does not contain the free boundary but still keeps its original form in the other subdomain. Therefore, the original variational inequality can be discretized separately with P_1 finite element in different subdomains. A non-overlapping domain decomposition method is introduced to solve these two discretized sub-problems by P_1 finite element method iteratively while a Robin type boundary condition is utilized for the data transfer on the common boundary. We show that the sequence of such finite element solutions converges to the discretized solution of the original problem. Application to a free seepage problem verifies the theory.

Keywords: Domain decomposition; Free boundary problem; Finite element method

1. Introduction

Free boundary value problems have been an important topic for mathematicians and engineers for a long time because many physical and engineering problems such as fluid flow in porous media, obstacle problems, elastic problems and lubrication phenomena fall under this category. These problems can usually be transformed into variational inequalities. Friedman [10] provided detailed theoretical analysis about regularity of the solution and the free boundary. Meanwhile, different kinds of numerical methods are proposed to solve general free boundary problems, see [3,6,8,11] and the references therein. Among those methods, finite element method is the most popular one. It is shown in [2,9] that $\lim_{h\to 0}\|u_h - u\| = 0$ where u_h and u are the finite element solution with mesh size h and the original analytic solution, respectively. Therefore, the finite element solution can be considered as a good approximation to the analytic solution of the variational inequality when the mesh size h is small enough.

In recent years, overlapping domain decomposition (DD) methods have been combined with finite element method to solve variational inequalities in order to improve computational performance under a parallel computing environment. The basic idea is to split the original domain into several overlapping subdomains and solve the variational inequality on each subdomain iteratively via data transfer from the common area between those subdomains. The authors in Refs. [4,1,12,16,18] and their references provide many variants of this approach. The key idea of these methods is to construct a sequence $\{u_h^n\}$ satisfying $\lim_{n\to\infty}\|u_h^n - u_h\|_1 = 0$ where $n = 1, 2, \ldots$ is the iteration step and h is the mesh size. The advantage of this approach is that u_h^n can be solved more efficiently on parallel computers while solving for u_h on the whole domain is

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very time and resource consuming, especially when the domain is irregular.

However, for many practical problems in the engineering and industrial fields, it is much easier and more convenient to split the original domain into two or more non-overlapping subdomains and then take care of the problems in each subdomain where the original problem may show different behavior. The non-overlapping DD method has been successfully applied for partial differential equations, see [7,14,15,17] and their references, but it has seldom been used to solve variational inequalities. In [13], we applied the non-overlapping DD method to solve the above variational inequality problem where the domain is split into two non-overlapping subdomains and the free boundary is only located in one subdomain. A robin boundary condition is utilized on the common boundary between these two subdomains. We show that at the continuous level $\lim_{n\to\infty} u^n = u$ where $\{u^n\}$ is a sequence of approximate analytic solutions based on the domain decomposition and u is the original analytic solution. Therefore, the non-overlapping DD method shows its great potential to solve variational inequality problems numerically, especially under a parallel environment.

In this paper, we will construct a discretized version of non-overlapping DD method to approximate u_h, the finite element solution, instead of the analytic solution u. That is, we will split the original domain into two non-overlapping subdomains and construct a sequence of finite element solution $\{u_h^n\}_{n=1}^{\infty}$ which will converge to u_h when $n \to \infty$.

This paper is organized as follows. In Section 2, the general free boundary problem is reformulated as a variational inequality and is discretized with conforming P_1 finite element method. A non-overlapping DD algorithm is then introduced for the construction of $\{u_h^n\}$ in different subdomains. In Section 3, convergence analysis of $\{u_h^n\}$ towards u_h is given. In Section 4, we provide the implementation details of the new DD method and apply this method to solve a free surface seepage problem. The numerical result confirms the theory. Conclusion of the paper and some future efforts are provided in Section 5.

2. Problem formulation

Let D be a domain in R^2, whose boundary will be denoted by ∂D, and $D = D_1 \cup D_2 \cup \Gamma_0$, D_1 and D_2 are open sets, $D_1 \cap D_2 = \phi$. $\Gamma_0 = \overline{D_1} \cap \overline{D_2}$ is the common boundary between D_1 and D_2. $\Gamma_1 = \partial D \cap \overline{D_1}$ and $\Gamma_2 = \partial D \cap \overline{D_2}$ represent the boundary of D (see Fig. 1). The general free boundary problem is formulated as follows:

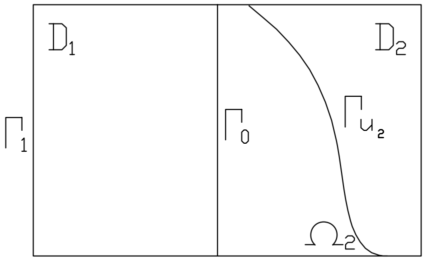

Fig. 1. The original free boundary problem.

Find an open set $\Omega_2 \subset D_2$ and $u(x) \in H^2(D)$ with $x = (x_1, x_2)$, such that

$$\begin{cases} -\Delta u + c(x)u = f(x) & \text{in } \Omega = D_1 \cup \Omega_2 \cup \Gamma_0, \\ (-\Delta u + c(x)u - f(x)) \cdot u(x) = 0 & \text{in } D, \\ u \geqslant 0 & \text{in } D, \\ -\Delta u + c(x)u - f(x) \geqslant 0 & \text{in } D, \\ u = g(x) & \text{on } \partial D, \\ D_2 - \Omega_2 = \{x \in D | u(x) = 0\}, \end{cases} \tag{2.1}$$

where $c(x) \geqslant 0$, $c(x) \in C^{\alpha}(\overline{D})$, $f(x) \in C^{\alpha}(\overline{D})$, $g(x) \in C^{2+\alpha}(\overline{D})$, $g(x) \geqslant 0$ on Γ_1 and $g(x) = 0$ on Γ_2, and ∂D is in $C^{2+\alpha}$. Here $C^{m+\alpha}(\overline{D})$ denotes the space of functions whose derivatives up to order m are Holder continuous with $0 < \alpha < 1$.

Friedman [10] showed that the free boundary Problem 2.1 is equivalent to the following variational inequality:

Find $u \in H = \{u : u \in H^1(D),\ u = g(x) \text{ on } \partial D,\ u \geqslant 0 \text{ in } D\}$, such that

$$a(u, v - u) \geqslant \langle f, v - u\rangle \quad \forall v \in H, \tag{2.2}$$

where

$$a(u, v) = \int_D \left[\sum_{i=1}^{2} \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i} + c(x)u(x)v(x)\right] \mathrm{d}x,$$

$$\langle f, v\rangle = \int_D f(x)v(x)\,\mathrm{d}x.$$

With u determined in D, we can define $\Omega_2 = \{(x_1, x_2) \in D_2 : u(x_1, x_2) > 0\}$.

Lemma 1 (Solution Uniqueness, [10]). *Problem (2.2) has a unique solution $u(x) \in W^{2,p}(D)$ for any $p < \infty$.*

However, there is no exact formula for the solution $u(x)$ of (2.1) or (2.2) generally. How to find its approximate solution efficiently is the concern of the computational scientists. Several different numerical methods have been developed for the numerical solution of (2.2) and finite element method is the most popular one because it can be implemented very easily and can achieve high-order precision.

To define the finite element solution of (2.2), we start with finite element partition of the domain D. Let Σ_h be a quasi-uniform triangular partition of D with mesh size h. Meanwhile, let N_h denote the set of all element nodes of Σ_h in the interior of D and Γ_h denote the element nodes of Σ_h on the boundary of D. Then $S_h \subset C^0(D)$, the conforming P_1 finite element space over Σ_h is defined as follows:

$$\begin{aligned} S_h = \{v : v|_{\tau} \in P_1(\tau)\ \forall \tau \in \Sigma_h, v \text{ is continuous at } p \in N_h, \\ v(p) \geqslant 0 \text{ at } p \in N_h \text{ and } v(p) = g(p) \text{ at } p \in \Gamma_h\}, \end{aligned} \tag{2.3}$$

where, if we can define $\{\phi_p\}_{p\in N_h}$ as the nodal basis functions of S_h with respect to N_h, then $v\in S_h$ can be written as

$$v=\sum_{p\in N_h} v(p)\phi_p. \tag{2.4}$$

In addition, for any $v\in S_h$, its discrete H_1 norm is defined as $\|v\|^2_{1,h}=\sum_{\tau\in\Sigma_h}\|v\|^2_{1,\tau}$ and its discrete semi-norm is defined similarly.

Based on the above notations, the finite element solution of (2.2) is shown below:

Find $u_h\in S_h$ such that

$$a_h(u_h,v_h-u_h)\geqslant\langle f,v_h-u_h\rangle\quad\forall v_h\in S_h, \tag{2.5}$$

where

$$\begin{aligned}a_h(u_h,v_h)&=\sum_{\tau\in\Sigma_h}a_\tau(u_h,v_h)\\&=\sum_{\tau\in\Sigma_h}\int_\tau\left[\sum_{i=1}^{2}\frac{\partial u_h(x)}{\partial x_i}\frac{\partial v_h(x)}{\partial x_i}+c(x)u_h(x)v_h(x)\right]\mathrm{d}x.\end{aligned} \tag{2.6}$$

It is shown in [2,9] that (2.5) has a unique solution which has the following error estimate:

Lemma 2. (Error Estimate, [2])

$$\|u_h-u\|_{1,h}=\mathrm{O}(h).$$

Lemma 2 states that the finite element solution u_h of (2.5) can be considered as a good approximation to the original analytic solution u. Therefore, how to obtain u_h efficiently is important and necessary. In [2,5,9], conforming P_1 finite element method is applied to solve (2.5) on the whole domain D, no matter how complicated the domain is.

However, we notice that the original problem shows different behavior in two subdomains D_1 and D_2 (see Fig. 1), i.e., the free boundary is only located in D_2. Therefore, it is reasonable to split the original problem into two sub-problems and solve them in D_1 and D_2 under a parallel computing environment, respectively. In this way, we can generate a more efficient numerical method than traditional finite element methods.

In [13], we proposed a new non-overlapping DD method to the original problem (2.1) and showed that the sequence $\{u^n(x)\}_{n=1}^\infty$ constructed by DD method converges to $u(x)$ when $n\to\infty$. The algorithm is shown as below where u_i^n and u_i stand for the restriction of $u^n(x)$ and $u(x)$ in D_i, $i=1,2$:

Algorithm 1

Let $g_1^1=g_2^1=0$ on Γ_0 and $n=1$.

Step 1. Solve $u_1^n\in H^1(D_1)$ and $u_2^n\in H^1(D_2)$ such that

$$a_1(u_1^n,v)+\int_{\Gamma_0}u_1^n v\,\mathrm{d}s=\langle f,v\rangle_1+\int_{\Gamma_0}g_1^n v\,\mathrm{d}s\qquad\forall v\in H^1(D_1),\quad v|_{\Gamma_1}=0, \tag{2.7a}$$

$$a_2(u_2^n,v-u_2^n)+\int_{\Gamma_0}u_2^n(v-u_2^n)\,\mathrm{d}s\geqslant\langle f,v-u_2^n\rangle_2+\int_{\Gamma_0}g_2^n(v-u_2^n)\,\mathrm{d}s\quad\forall v\in H_2. \tag{2.7b}$$

Step 2. Define

$$g_2^{n+1}=2u_1^n-g_1^n,\quad g_1^{n+1}=2u_2^n-g_2^n, \tag{2.7c}$$

and repeat Step 1 with n replaced by $n+1$.

Lemma 3. (Continuous Convergence, [13])

$$\lim_{n\to\infty}\|u_1^n-u_1\|_{1,D_1}=\lim_{n\to\infty}\|u_2^n-u_2\|_{1,D_2}=0.$$

Lemma 3 shows that the non-overlapping DD method is very promising in solving free boundary value problems, at least at the continuous level. In this paper, we will construct a new sequence of discrete functions $\{u_h^n\}_{n=1}^\infty\subset S_h$ in order to approximate the finite element solution u_h of (2.3). To this end, we need to investigate the behavior of u_h in D_1 and D_2 as the basis for the construction of u_h^n. For simplicity, assume the finite element mesh coincide with the common boundary Γ_0 between D_1 and D_2 so that no triangle crosses over Γ_0, see Fig. 2. This can be realized easily so that Σ_h restricted in each subdomain is still a quasi-uniform finite element mesh. Let Σ_{1h} and Σ_{2h} denote the restriction of Σ_h on $D_1\cup\Gamma_0$ and $D_2\cup\Gamma_0$, N_{1h} and N_{2h} denote the sets of element nodes of Σ_{1h} and Σ_{2h}, respectively. Then N_{1h} and N_{2h} both contain the element nodes on Γ_0 in addition to the nodes in their subdomains. $S_{ih}\subset C^0(D_i)$ is the conforming P_1 finite element space over Σ_{ih}, correspondingly. For any $v\in S_{ih}$, its discrete H^1 norm is defined as $\|v\|^2_{1,D_i,h}=\sum_{\tau\in\Sigma_{ih}}\|v\|^2_{1,\tau}$ and the same for its discrete semi-norm.

Let Γ_{0h} denote the element nodes of Σ_h on Γ_0, and for each $p\in\Gamma_{0h}$, define

$$\begin{cases}G_1^p=-a_{2h}(u_{2,h},\phi_p)+\langle f,\phi_p\rangle_2,\\G_2^p=-a_{1h}(u_{1,h},\phi_p)+\langle f,\phi_p\rangle_1,\end{cases} \tag{2.8}$$

where $u_{i,h}$ is the restriction of u_h in $\overline{D_i}$, $a_{ih}(u,v)=\sum_{\tau\in\Sigma_{ih}}a_\tau(u,v)$, and $\langle f,v\rangle_i=\sum_{\tau\in\Sigma_{ih}}\int_\tau fv\,\mathrm{d}x$, $i=1,2$. $\{G_1^p\}_{p\in\Gamma_{0h}}$ and $\{G_1^p\}_{p\in\Gamma_{0h}}$ are then two real number arrays whose dimensions are equal to the number of element nodes on Γ_0.

Lemma 4. *For any $v\in S_{1h}$ satisfying $v|_{\Gamma_1}=0$, $u_{1,h}$ satisfies*

$$a_{1h}(u_{1,h},v)=\langle f,v\rangle_1+\sum_{p\in\Gamma_{0h}}G_1^p v(p).$$

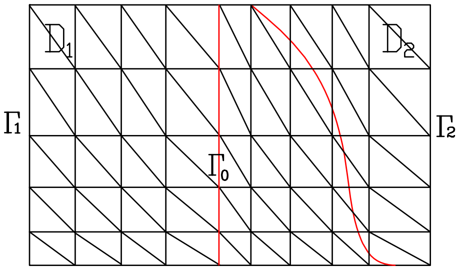

Fig. 2. Finite element meshes.

Proof. Since the free boundary is located in D_2 only and $u_{1,h} = u_h|_{D_1}$, then $u_{1,h}(p) > 0$ for any $p \in N_{1h}$, which is the set of element nodes on $D_1 \cup \Gamma_0$. Therefore, for given $v \in S_{1h}$ with $v|_{\Gamma_1} = 0$, we can always choose $\alpha > 0$ which is so small that

$$(u_{1,h} + \alpha v)(p) > 0 \quad \text{and} \quad (u_{1,h} - \alpha v)(p) > 0 \quad \forall p \in N_{1h}$$

hold at the same time. We then define

$$v_h = \begin{cases} u_{1,h} + \alpha v & \text{in } \overline{D_1}, \\ u_{2,h} & \text{in } D_2 \end{cases} \tag{2.9}$$

and it is obvious that $v_h \in S_h$. Putting v_h into (2.5), and noticing that $u_{1h} = u_{2h} = u_h$ on Γ_0, we have

$$\begin{aligned} &a_{1h}(u_{1,h}, \alpha v) + \alpha \sum_{p \in \Gamma_{0h}} a_{2h}(u_{2,h}, \phi_p) v(p) \\ &\quad \geqslant \langle f, \alpha v \rangle_1 + \alpha \sum_{p \in \Gamma_{0h}} \langle f, \phi_p \rangle_2 v(p), \end{aligned}$$

where $a_{2h}(u_{2,h}, v_h - u_{2,h})$ is reduced to the second term of the left hand side since (2.9) implies that in D_2,

$$(v_h - u_{2,h})(p) = \begin{cases} 0 & p \in N_{1h} - \Gamma_{0h}, \\ \alpha v(p) & p \in \Gamma_{0h}, \end{cases}$$

and $\langle f, v_h - u_{2,h} \rangle$ is reduced to the second term on the right hand for the same reason. Therefore, dividing the last inequality by α yields

$$a_{1h}(u_{1,h}, v) + \sum_{p \in \Gamma_{0h}} a_{2h}(u_{2,h}, \phi_p) v(p) \geqslant \langle f, v \rangle_1 + \sum_{p \in \Gamma_{0h}} \langle f, \phi_p \rangle_2 v(p). \tag{2.10}$$

Similarly, if we define

$$v_h = \begin{cases} u_{1,h} - \alpha v & \text{in } \overline{D_1}, \\ u_{2,h} & \text{in } D_2 \end{cases} \tag{2.11}$$

and put it into (2.5), we obtain

$$\begin{aligned} &a_{1h}(u_{1,h}, v) + \sum_{p \in \Gamma_{0h}} a_{2h}(u_{2,h}, \phi_p) v(p) \\ &\quad \leqslant \langle f, v \rangle_1 + \sum_{p \in \Gamma_{0h}} \langle f, \phi_p \rangle_2 v(p). \end{aligned} \tag{2.12}$$

Combination of (2.10) and (2.12) finishes the proof. □

Lemma 4 implies that $u_{1,h}$ satisfies a PDE in D_1 where there is no free boundary. However, $u_{2,h}$ should satisfy a variational inequality in D_2 which contains the free boundary.

Lemma 5. *For any $v \in S_{2h}$, $u_{2,h}$ satisfies*

$$a_{2h}(u_{2,h}, v - u_{2,h}) \geqslant \langle f, v - u_{2,h} \rangle_2 + \sum_{p \in \Gamma_{0h}} G_2^p (v - u_{2,h})(p).$$

Proof. Define

$$v_h = \begin{cases} v & \text{in } \overline{D_2}, \\ u_{1,h} & \text{in } D_1, \end{cases} \tag{2.13}$$

then $v \in S_h$. Putting it into (2.5), and noticing that $u_{1h} = u_{2h} = u_h$ on Γ_0, we have

$$\begin{aligned} &a_{2h}(u_{2,h}, v - u_{2,h}) + \sum_{p \in \Gamma_{0h}} a_{1h}(u_{1,h}, \phi_p)(v - u_{2,h})(p) \\ &\quad \geqslant \langle f, v - u_{2,h} \rangle_2 + \sum_{p \in \Gamma_{0h}} \langle f, \phi_p \rangle_1 (v - u_{2,h})(p), \end{aligned}$$

i.e.,

$$a_{2h}(u_{2,h}, v - u_{2,h}) \geqslant \langle f, v - u_{2,h} \rangle_2 + \sum_{p \in \Gamma_{0h}} G_2^p (v - u_{2,h})(p). \quad \square$$

From Lemmas 4 and 5, we can see that $u_{1,h}$ and $u_{2,h}$ show different behaviors in D_1 and D_2 where an additional term involving either G_1^p or G_2^p appears. The following Lemma will reflect the relationship between G_1^p and G_2^p for $p \in \Gamma_{0h}$.

Lemma 6. *For each $p_0 \in \Gamma_{0h}$,*

$$G_1^{p_0} = -G_2^{p_0}.$$

Proof. The proof follows the same logic as Lemma 4. Since $u_h(p) > 0$ for any $p \in N_{1h}$, for the given ϕ_{p_0}, we can always choose α so small that $(u_h \pm \alpha \phi_{p_0})(p) > 0$ for any $p \in N_{1h}$. If we define $v_h = u_h + \alpha \phi_{p_0}$ then $v_h \in S_h$. By inserting it into (2.5), we obtain

$$a_{1h}(u_{1,h}, \alpha \phi_{p_0}) + a_{2h}(u_{2,h}, \alpha \phi_{p_0}) \geqslant \langle f, \alpha \phi_{p_0} \rangle_1 + \langle f, \alpha \phi_{p_0} \rangle_2,$$

i.e.,

$$a_{1h}(u_{1,h}, \phi_{p_0}) + a_{2h}(u_{2,h}, \phi_{p_0}) \geqslant \langle f, \phi_{p_0} \rangle_1 + \langle f, \phi_{p_0} \rangle_2. \tag{2.14}$$

Similarly, choosing $v_h = u_h - \alpha \phi_{p_0}$ yields

$$a_{1h}(u_{1,h}, \phi_{p_0}) + a_{2h}(u_{2,h}, \phi_{p_0}) \leqslant \langle f, \phi_{p_0} \rangle_1 + \langle f, \phi_{p_0} \rangle_2. \tag{2.15}$$

Combination of (2.14) and (2.15) gives

$$a_{1h}(u_{1,h}, \phi_{p_0}) + a_{2h}(u_{2,h}, \phi_{p_0}) = \langle f, \phi_{p_0} \rangle_1 + \langle f, \phi_{p_0} \rangle_2,$$

i.e.,

$$G_1^{p_0} = -G_2^{p_0}. \quad \square$$

By comparing Lemmas 4, 5 with Algorithm 1, we found that Lemma 4 has the same structure as (2.7a) of Algorithm 1 while Lemma 5 has the same structure as (2.7b) of Algorithm 1. Therefore, based on Lemmas 4–6 which relate the additional terms in Lemmas 4 and 5, we propose the following discrete non-overlapping DD method in order to construct two sequences $\{u_{1,h}^n\}_{n=1}^{\infty} \subset S_{1h}$ and $\{u_{2,h}^n\}_{n=1}^{\infty} \subset S_{2h}$ to approximate $u_{1,h}$ and $u_{2,h}$ as follows:

Algorithm 2

Let $g_1^1(p) = g_2^1(p) = 0$ for $p \in \Gamma_{0h}$ and $n = 1$.

Step 1. Solve $u_{1,h}^n \in S_{1h}$ and $u_{2,h}^n \in S_{2h}$ such that

$$\begin{aligned} &a_{1h}(u_{1,h}^n, v) + \sum_{p \in \Gamma_{0h}} u_{1,h}^n(p) v(p) \\ &\quad = \langle f, v \rangle_1 + \sum_{p \in \Gamma_{0h}} g_1^n(p) v(p) \quad \forall v \in S_{1h}, \ v|_{\Gamma_1} = 0, \end{aligned} \tag{2.16a}$$

$$a_{2h}(u^n_{2,h}, v-u^n_{2,h}) + \sum_{p\in\Gamma_{0h}} u^n_{2,h}(p)(v-u^n_{2,h})(p) \geqslant \langle f, v-u^n_{2,h}\rangle_2 + \sum_{p\in\Gamma_{0h}} g^n_2(p)(v-u^n_{2,h})(p) \quad \forall v\in S_{2h}. \tag{2.16b}$$

Step 2. Define

$$\begin{aligned} g^{n+1}_2(p) &= 2u^n_{1,h}(p) - g^n_1(p),\\ g^{n+1}_1(p) &= 2u^n_{2,h}(p) - g^n_2(p) \quad \text{for } p\in\Gamma_{0,h} \end{aligned} \tag{2.16c}$$

and repeat Step 1 with n replaced by $n+1$.

The $\sum_{p\in\Gamma_{0h}}$ term from Algorithm 2 can be considered as the discrete version of the $\int_{\Gamma_0}$ term from Algorithm 1. It will be shown in Section 4 that the easy form of $\sum_{p\in\Gamma_{0h}}$ term can greatly reduce the computational burden by avoiding numerical integration on Γ_0. In Section 3, we will show that $\lim_{n\to\infty} u^n_{i,h} = u_{i,h}, i=1,2$.

3. Convergence analysis of non-overlapping DD method

In order to show that $u^n_{i,h}\to u_{i,h}$ as $n\to\infty$, we need to investigate the relationship between g^n_i terms associated with $u^n_{i,h}$ from Algorithm 2 and G^p_i terms associated with $u_{i,h}$ from Lemmas 4 and 5.

Define $e^n_{i,h} = u^n_{i,h} - u_{i,h}$ in D_i, and $\widetilde{g}^n_i(p) = g^n_i(p) - (u_{i,h}(p) + G^p_i)$ for $p\in\Gamma_{0h}$. Then for $p\in\Gamma_{0h}$, we have

$$\begin{aligned} \widetilde{g}^{n+1}_2(p) &= g^{n+1}_2(p) - (u_{2,h}(p) + G^p_2)\\ &= 2u^n_{1,h}(p) - g^n_1(p) - u_{2,h}(p) - G^p_2\\ &= 2u^n_{1,h}(p) - 2u_{1,h}(p) - g^n_1(p) + u_{1,h}(p) + G^p_1\\ &= 2(u^n_{1,h}(p) - u_{1,h}(p)) - [g^n_1(p) - (u_{1,h}(p) + G^p_1)]\\ &= 2e^n_{1,h}(p) - \widetilde{g}^n_1(p) \end{aligned} \tag{3.1}$$

by applying (2.16c), the fact that $u_{1,h}(p) = u_{2,h}(p)$ on Γ_0, and Lemma 6, respectively.

Similarly,

$$\widetilde{g}^{n+1}_1(p) = 2e^n_{2,h}(p) - \widetilde{g}^n_2(p) \quad \text{for } p\in\Gamma_{0h}. \tag{3.2}$$

Therefore, if we define

$$\|\widetilde{g}^{n+1}\|^2_{\Gamma_0} = \sum_{p\in\Gamma_{0h}} \left[|\widetilde{g}^{n+1}_1(p)|^2 + |\widetilde{g}^{n+1}_2(p)|^2\right],$$

we have

$$\begin{aligned} \|\widetilde{g}^{n+1}\|^2_{\Gamma_0} &= \sum_{p\in\Gamma_{0h}} [2e^n_{1,h}(p) - \widetilde{g}^n_1(p)]^2 + \sum_{p\in\Gamma_{0h}} [2e^n_{2,h}(p) - \widetilde{g}^n_2(p)]^2\\ &= \|\widetilde{g}^n\|^2_{\Gamma_0} + 4\sum_{p\in\Gamma_{0h}} (e^n_{1,h}(p) - \widetilde{g}^n_1(p))e^n_{1,h}(p)\\ &\quad + 4\sum_{p\in\Gamma_{0h}} (e^n_{2,h}(p) - \widetilde{g}^n_2(p))e^n_{2,h}(p). \end{aligned} \tag{3.3}$$

Furthermore, subtraction of (2.16a) by the equation in Lemma 4 yields

$$a_{1h}(e^n_{1,h}, v) + \sum_{p\in\Gamma_{0h}} u^n_{1,h}(p)v(p) = \sum_{p\in\Gamma_{0h}} (g^n_1(p) - G^p_1)v(p) \quad \forall v\in S_{1h}, \quad v|_{\Gamma_1} = 0,$$

i.e.,

$$a_{1h}(e^n_{1,h}, v) = \sum_{p\in\Gamma_{0h}} (\widetilde{g}^n_1(p) - e^n_{1,h}(p))v(p) \quad \forall v\in S_{1h}, \quad v|_{\Gamma_1} = 0. \tag{3.4}$$

Since $e^n_{1,h} = u^n_{1,h} - u_{1,h}\in S_{1h}$ and $e^n_{1,h}|_{\Gamma_1} = 0$, we replace v by $e^n_{1,h}$ in (3.4) and obtain

$$\sum_{p\in\Gamma_{0h}} (e^n_{1,h}(p) - \widetilde{g}^n_1(p))e^n_{1,h}(p) = -a_{1h}(e^n_{1,h}, e^n_{1,h}). \tag{3.5}$$

Finally, let us estimate $e^n_{2,h}$ in D_2. Since $u^n_{2,h}\in S_{2h}$, we can replace v by $u^n_{2,h}$ in Lemma 5 and have

$$a_{2h}(u^n_2, u^n_{2,h} - u^n_2) \geqslant \langle f, u^n_{2,h} - u^n_2\rangle_2 + \sum_{p\in\Gamma_{0h}} G^p_2(u^n_{2,h} - u_{2,h})(p). \tag{3.6}$$

Similarly, since $u_{2,h}\in S_{2h}$, by replacing v by $u_{2,h}$ in (2.16b) of Algorithm 2, we obtain

$$a_{2h}(u^n_{2,h}, u^n_2 - u^n_{2,h}) + \sum_{p\in\Gamma_{0h}} u^n_{2,h}(p)(u_{2,h} - u^n_{2,h})(p) \geqslant \langle f, u^n_2 - u^n_{2,h}\rangle_2 + \sum_{p\in\Gamma_{0h}} g^n_2(p)(u_{2,h} - u^n_{2,h})(p). \tag{3.7}$$

Adding (3.6) and (3.7) yields

$$\begin{aligned} -a_{2h}(e^n_{2,h}, e^n_{2,h}) &\geqslant \sum_{p\in\Gamma_{0h}} (G^p_2 + u^n_{2,h}(p) - g^n_2(p))e^n_{2,h}(p)\\ &= \sum_{p\in\Gamma_{0h}} (e^n_{2,h}(p) - \widetilde{g}^n_2(p))e^n_{2,h}(p), \end{aligned}$$

i.e.,

$$\sum_{p\in\Gamma_{0h}} (e^n_{2,h}(p) - \widetilde{g}^n_2(p))e^n_{2,h}(p) \leqslant -a_{2h}(e^n_{2,h}, e^n_{2,h}). \tag{3.8}$$

By replacing the last two terms on the right hand side of (3.3) with (3.5) and (3.8), we have

$$\|\widetilde{g}^{n+1}\|^2_{\Gamma_0} \leqslant \|\widetilde{g}^n\|^2_{\Gamma_0} - 4a_{1h}(e^n_{1,h}, e^n_{1,h}) - 4a_{2h}(e^n_{2,h}, e^n_{2,h}). \tag{3.9}$$

From (3.9),

$$a_{1h}(e^n_{1,h}, e^n_{1,h}) + a_{2h}(e^n_{2,h}, e^n_{2,h}) \leqslant \frac{1}{4}(\|\widetilde{g}^n\|^2_{\Gamma_0} - \|\widetilde{g}^{n+1}\|^2_{\Gamma_0}). \tag{3.10}$$

Summation of (3.10) from $n=1$ to N yields

$$\sum_{n=1}^{N}\left[a_{1h}(e^n_{1,h}, e^n_{1,h}) + a_{2h}(e^n_{2,h}, e^n_{2,h})\right] \leqslant \frac{1}{4}(\|\widetilde{g}^1\|^2_{\Gamma_0} - \|\widetilde{g}^N\|^2_{\Gamma_0}) \leqslant \frac{1}{4}M.$$

This holds for arbitrarily large N, where $M = \|\widetilde{g}^1\|^2_{\Gamma_0}$ is a constant. Therefore,

$$\lim_{n\to\infty}[a_{1h}(e^n_{1,h}, e^n_{1,h}) + a_{2h}(e^n_{2,h}, e^n_{2,h})] = 0.$$

Since both terms in the above limit are non-negative, we have

$$\lim_{n\to\infty} a_{1h}(e^n_{1,h}, e^n_{1,h}) = 0 \tag{3.11}$$

and

$$\lim_{n\to\infty} a_{2h}(e^n_{2,h}, e^n_{1,D_2,h}) = 0. \tag{3.12}$$

To prove $\|e^n_{1,h}\|_{1,h} \to 0$, we should consider the following two cases:

Case 1. $c(x) \geqslant C_0 > 0$, then

$$a_{1h}(e^n_{1,h}, e^n_{1,h}) = \sum_{\tau\in\Sigma_{1h}} \left[\int_\tau \nabla e^n_{1,h} \nabla e^n_{1,h}\,\mathrm{d}x + C_0 \int_\tau e^n_{1,h} e^n_{1,h}\,\mathrm{d}x\right] \geqslant \min(1, C_0)\|e^n_{1,h}\|^2_{1,D_1,h}.$$

Then (3.11) yields $\|e^n_{1,h}\|_{1,D_1,h} \to 0$. Similarly, $\|e^n_{2,h}\|_{1,D_2,h} \to 0$. This verifies the convergence of Algorithm 2.

Case 2. $c(x) \geqslant 0$. Then

$$a_{1,h}(e^n_{1,h}, e^n_{1,h}) \geqslant |e^n_{1,h}|^2_{1,D_1,h}.$$

Eq. (3.11) yields only $|e^n_{1,h}|_{1,D_1,h} \to 0$. Similarly, $|e^n_{2,h}|_{1,D_2,h} \to 0$. However, since $e^n_{1,h}|_{\Gamma_1} = 0$, the inverse inequality will imply $\|e^n_{1,h}\|_{1,D_1,h} \to 0$. Similarly, $\|e^n_{2,h}\|_{1,D_2,h} \to 0$. This also makes Algorithm 2 convergent. Combining the above two cases, we obtain the following theorem for the convergence of non-overlapping DD method.

Theorem 7. *Suppose* $\{u^n_{1,h}\}$, $\{u^n_{2,h}\}$ *are obtained from Algorithm 2,* $u_{1,h}$ *and* $u_{2,h}$ *are the restriction of* u_h *in* D_1 *and* D_2, *then*

$$\lim_{n\to\infty} \|u^n_{1,h} - u_{1,h}\|_{1,D_1,h} = \lim_{n\to\infty} \|u^n_{2,h} - u_{2,h}\|_{1,D_2,h} = 0. \tag{3.13}$$

Theorem 7 shows that when the domain is complicated, we can split the domain into two or more non-overlapping subdomains and apply the finite element method to solve the corresponding discrete sub-problems iteratively. The sequence of the finite element solutions will converge to the original discrete solution u_h. It provides us an easy way to approximate u_h in a parallel computing environment. In Section 4, we will provide the implementation details of Algorithm 2 and apply the method to solve a free seepage problem.

4. Numerical example

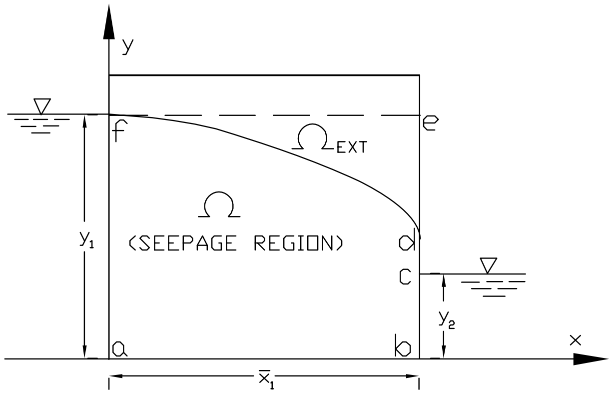

Fig. 3. Free boundary seepage problem.

In this Section, we will provide the implementation details of the non-overlapping DD Algorithm 2. That is, how to transform (2.16a) and (2.16b) into an explicit formula which can compute $u^n_{i,h}$ values at both the internal element nodes and the nodes on Γ_0. To clarify the idea, we consider the problem of free surface seepage as a working example (see Fig. 3): find the free surface in a steady, two-dimensional seepage through a rectangular dam. In this study, the free surface, whose position is not known in advance, is to be found. In the seepage region Ω with $(x_1, x_2) = (x, y)$, the velocity potential ϕ must satisfy the following:

$$\begin{aligned} &\Delta\phi = 0 && \text{in } \Omega,\\ &\phi = y_1 && \text{on } [af],\\ &\phi = y_2 && \text{on } [bc],\\ &\phi = y && \text{on } [cd],\\ &\phi = y && \text{on } \widehat{fd},\\ &\phi_n = 0 && \text{on } \widehat{fd},\\ &\phi_y = 0 && \text{on } [ab], \end{aligned} \tag{4.1}$$

where y_1 and y_2 are the heights of the water on the left and right sides, respectively and n is the outward normal direction of $\widehat{fd}$. The flow domain Ω is not known since the location of the free surface $\widehat{fd}$ needs to be found. Define D to be $D = \{(x, y) : 0 < x < \bar{x}_1, 0 < y < y_1\}$ and define $\overline{\phi}$ as an extension of ϕ as follows:

$$\overline{\phi} = \begin{cases} \phi(x, y) & \text{in } \Omega,\\ y & \text{in } \overline{D} - \Omega = \Omega_{\text{ext}}. \end{cases} \tag{4.2}$$

Using the Baiocchi transformation, a new variable is defined as

$$w(x, y) = \int_y^{y_1} [\overline{\phi}(x, \eta) - \eta]\,\mathrm{d}\eta. \tag{4.3}$$

Then w satisfies

$$\begin{aligned} &\Delta w = \chi_\Omega && \text{in } D,\\ &w = 0 && \text{in } \overline{D} - \overline{\Omega}, \end{aligned} \tag{4.4}$$

with the associated boundary conditions

$$\begin{aligned} &w(0, y) = \frac{1}{2}(y_1 - y)^2 && \text{on } [af],\\ &w(x, 0) = \frac{y_1^2}{2} - \frac{y_1^2 - y_2^2}{2\bar{x}_1}x && \text{on } [ab],\\ &w(\bar{x}_1, y) = \frac{1}{2}(y_2 - y)^2 && \text{on } [bc],\\ &w(x, y) = 0 && \text{on } [ce] \cup [fe]. \end{aligned} \tag{4.5}$$

Eqs. (4.4) and (4.5) are equivalent to the following variational inequality problem:

Find $w(x,y) \in H = \{u : u \in H^1(D),\ u|_{\partial D}$ satisfies (4.5)$\}$ such that

$$\int_D \nabla w \cdot \nabla(v - w)\,\mathrm{d}x\,\mathrm{d}y \geqslant \int_D -(v - w)\,\mathrm{d}x\,\mathrm{d}y \quad \forall v \in H. \tag{4.6}$$

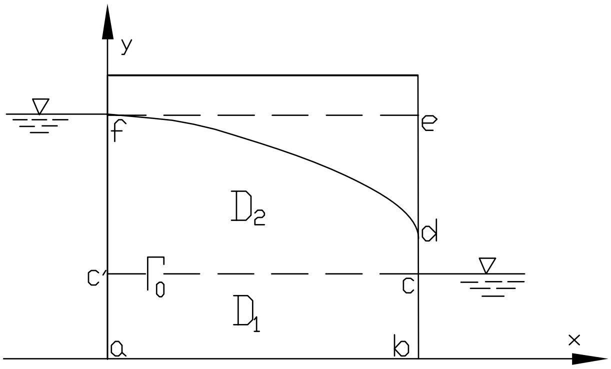

Fig. 4. Domain decomposition.

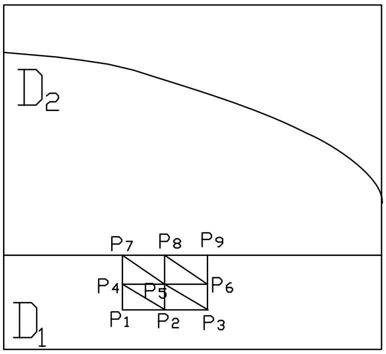

Fig. 5. Triangulation.

If we can solve (4.6) for w, the following quantities can be obtained:

$$\Omega = \{(x,y) : (x,y) \in D, w(x,y) > 0\},$$
$$\phi = y - w_y \quad \text{in } \Omega.$$

In order to apply the non-overlapping Algorithm 2 to solve (4.6), we decompose D into two non-overlapping regions D_1 and D_2 with common boundary Γ_0 (see Fig. 4) such that D_2 is the region containing the free surface. The sample seepage problem has the following data: $y_1 = 1.00$, $y_2 = \frac{1}{6}$, $\bar{x}_1 = \frac{2}{3}$. The triangulation with uniform horizontal and vertical mesh size $\Delta x = \Delta y = h = 0.0069$ is applied to the whole domain D, where each small square of size h is split diagonally from upper left to lower right into two triangles. Fig. 5 only shows part of the triangulation in D_1 where P_1, P_2, P_3, P_4, P_5, P_6 are the internal mesh nodes and P_7, P_8, P_9 are the nodes on Γ_0. When applying Algorithm 2, we are concerned with solving (2.16a) and (2.16b) iteratively. Since S_{1h} and S_{2h} have finite dimensions, we apply the relaxation method for the variational inequality [11, Chapter 5] here for (2.16a) and (2.16b). The main idea of the relaxation method is to consider all the mesh nodes P_i sequentially at each step n. Let us consider D_1 here and we can take care of D_2 in the same way. As we go through each P_i which is either an internal mesh node in D_1 or a node on Γ_0, we need to solve the following (4.7a) for $u^n_{1,h}(P_i)$ when P_i is an internal node or solve (4.7b) when P_i is on Γ_0. When we solve $u^n_{1,h}(P_i)$, all the $u^n_{1,h}$ values at other nodes are temporarily fixed. Then we use the projected value $\max(u^n_{1,h}(P_i), 0)$ as the final choice of $u^n_{1,h}(P_i)$.

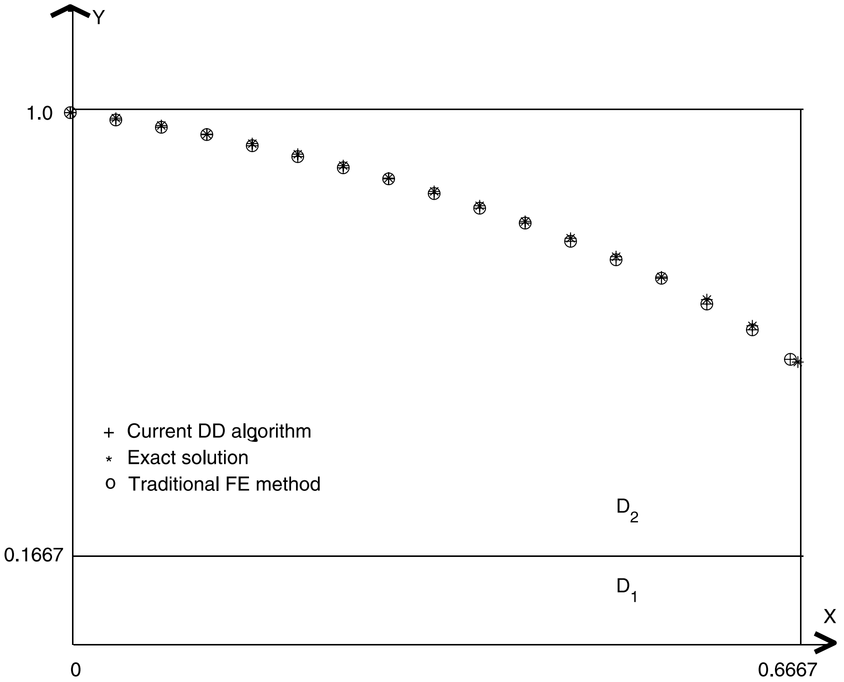

Fig. 6. Numerical result.

$$\int_{D_1} \nabla u^n_{1,h} \cdot \nabla \phi_{P_i} \,\mathrm{d}x\,\mathrm{d}y = \int_{D_1} -\phi_{P_i} \,\mathrm{d}x\,\mathrm{d}y, \tag{4.7a}$$

$$\int_{D_1} \nabla u^n_{1,h} \cdot \nabla \phi_{P_i} \,\mathrm{d}x\,\mathrm{d}y + u^n_{1,h}(P_i) = \int_{D_1} -\phi_{P_i} \,\mathrm{d}x\,\mathrm{d}y + g^n_1(P_i), \tag{4.7b}$$

when P_i is an internal mesh node, for example P_5, (4.7a) can be simplified as

$$u^n_{1,h}(P_5) = \frac{1}{4}[u^n_{1,h}(P_2) + u^n_{1,h}(P_4) + u^n_{1,h}(P_6) + u^n_{1,h}(P_8) - h^2], \tag{4.8a}$$

and when P_i is on Γ_0, for example, P_8, (4.7b) becomes

$$u^n_{1,h}(P_8) = \frac{1}{6}[u^n_{1,h}(P_7) + u^n_{1,h}(P_9) + 2u^n_{1,h}(P_5) + 2g^n_1(P_8) - h^2], \tag{4.8b}$$

convergence on D_1 and D_2 is determined when

$$\max_{P_i} |u^{n+1}_{1,h}(P_i) - u^n_{1,h}(P_i)| < \epsilon \quad \text{and}$$

$$\max_{P_i} |u^{n+1}_{2,h}(P_i) - u^n_{2,h}(P_i)| < \epsilon,$$

respectively, where $\epsilon = 1 \times 10^{-5}$ is some fixed positive constant.

The free surface is taken as the first mesh point with a value of w that is less than ϵ when you move in the vertical direction for a fixed x. The final free surface location is shown in Fig. 6 along with the numerical result by the traditional finite element method as in [11] and the exact solution reported by Crank [6], attributed to Polubarinova-Kochina. We can see that the new DD method is as good as the traditional finite element method in approximating the exact solution of the seepage problem. The total number of iteration steps to reach the tolerance by our DD method is 47, while the traditional finite element method requires 60 steps to reach the same tolerance.

5. Conclusion and future work

In this paper, we studied the finite element solution of variational inequalities arising from free boundary problems. We applied a non-overlapping DD method to split the original domain into two subdomains where the free boundary is only located in one subdomain and thus in the other subdomain only a partial differential equation needs to be considered. This domain decomposition technique generates a sequence of discrete functions which converges to the finite element solution of the problem. Each function from this sequence can be obtained more efficiently by parallel computation than the original finite element solution. Therefore, our DD technique improves numerical performance for general free boundary problems.

In our future work, we will apply popular non-matching grid techniques, especially mortar finite element method, to solve general free boundary problems and the associated variational inequalities. Since the major concern in this field is the determination of the free boundary, we will apply the fine grid in the neighborhood of the free boundary while apply the coarse grid away from the free boundary. This effort can achieve better precision for the free boundary while still maintain computational efficiency on the whole domain.

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