

# A Nonoverlapping Domain Decomposition Method for Variational Inequalities Derived from Free Boundary Problems

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The article proposes a nonoverlapping domain decomposition method for variational inequalities derived from free boundary problems. The free boundary value problem is broken up into two problems on nonoverlapping regions. In one region the problem is treated as a partial differential equation, while in the second region that contains the free boundary part, a variational inequality is considered. By solving these two related problems successively, we have shown that the successive solutions converge to the solution of the original problem. Application to a free surface seepage problem is given. © 2005 Wiley Periodicals, Inc. *Numer Methods Partial Differential Eq* 22: 1–17, 2006

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## I. INTRODUCTION

Domain decomposition (DD) methods have been intensively studied for partial differential equations. The first original work by Schwarz proposed an overlapping method for the solution of classical boundary value problems for harmonic functions. Afterwards, many researchers have conducted research along this direction. See [1] and [2] and the references therein for the research on overlapping domain decomposition methods for partial differential equations.

Meanwhile, nonoverlapping DD methods are also being investigated by researchers. Compared with the overlapping DD methods, the nonoverlapping DD methods have many advantages, such as direct split of the original domain into mutually disjoint subdomains where

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parallel computation can be applied in each subdomain. See [3]–[7] for all the algorithms and convergence analysis provided therein.

Problems described by variational inequalities have been an important topic for mathematicians and engineers for a long time because many physical and engineering problems such as fluid flow in porous media, obstacle problems, elastic problems, and lubrication phenomena fall under this category. These problems are in general of free boundary type and can be transformed into variational inequalities. Crank [8] and Elliott and Ockendon [9] proposed several numerical techniques to solve general free boundary problems. Friedman [10] provided detailed theoretical analysis about regularity of the solution and the free boundary.

Because domain decomposition methods have shown great advantage in the field of scientific computation, mathematicians began to apply the overlapping domain decomposition methods to solve variational inequality problems. The basic idea is to split the original domain into several overlapping subdomains and solve the variational inequality on each sub domain via data transfer from the common area between those subdomains. Bruch and Sloss [11], Tai et al. [12], Tai [13], Hoffmann and Zou [14, 15], Badea and Wang [16], Zeng and Zhou [17] and their references provide many variants of this approach, whereas convergence analyses of the algorithms and their application to many problems in different fields are provided.

Actually for many practical problems in the engineering and industrial fields, it is much easier and more convenient to split the original domain into two or three nonoverlapping subdomains and then take care of the problems in each subdomain where the original problem may show different behavior. However, no active research has been reported by applying nonoverlapping DD method for variational inequalities. Even though many numerical tests have shown the stability and convergence of nonoverlapping DD methods for variational inequalities, see Bruch [18], Papadopoulos et al. [19], Jiang et al. [20, 21], no convergence analysis has been fully provided.

In this article, we propose a free boundary problem that can be generalized into a variational inequality. We follow along the work of Lions [3] and Deng [6] to construct an algorithm to handle the original problem in nonoverlapping subdomains. A robin boundary condition is utilized on the common boundary between these two subdomains. We obtain the convergence result for this new method for our variational inequality problem.

This article is organized as follows. In Section 2, we reformulate the general free boundary problem as a two subdomain problem where the partial differential equation and the variational inequality are considered in each subdomain, respectively. In Section 3, convergence analysis of the domain decomposition method is provided. In Section 4, we apply our new nonoverlapping DDM to a free surface seepage problem and the numerical result confirms the convergence property. A summary of the article and some future considerations are outlined in Section 5.

## II. FORMULATION OF THE PROBLEM

Let  $D$  be a domain in  $R^2$ , whose boundary will be denoted by  $\partial D$ , and  $D = D_1 \cup D_2 \cup \Gamma_0$ ,  $D_1$  and  $D_2$  are open sets,  $D_1 \cap D_2 = \phi$ .  $\Gamma_0 = \overline{D_1} \cap \overline{D_2}$  is the common boundary between  $D_1$  and  $D_2$ .  $\Gamma_1 = \partial D \cap \overline{D_1}$  and  $\Gamma_2 = \partial D \cap \overline{D_2}$  represent the boundary of  $D$  (see Fig. 1).

The general free boundary problem is the following:

**Problem 2.1.** Find an open set  $\Omega_2 \subset D_2$  and  $u(x) \in H^2(D)$  with  $x = (x_1, x_2)$ , such that

$$-\Delta u + c(x)u = f(x) \quad \text{in } \Omega = D_1 \cup \Omega_2 \cup \Gamma_0; \quad (2.1)$$

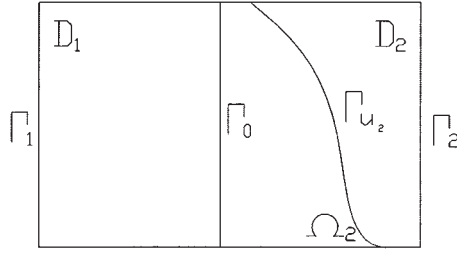


FIG. 1. The original free boundary problem.

$$(-\Delta u + c(x)u - f(x)) \cdot u(x) = 0 \quad \text{in } D; \tag{2.2a}$$

$$u \geq 0 \quad \text{in } D; \tag{2.2b}$$

$$-\Delta u + c(x)u - f(x) \geq 0 \quad \text{in } D; \tag{2.2c}$$

$$u = g(x) \quad \text{on } \partial D; \tag{2.3}$$

$$D_2 - \Omega_2 = \{x \in D \mid u(x) = 0\}, \tag{2.4}$$

where  $c(x) \geq 0$ ,  $c(x) \in C^\alpha(\bar{D})$ ,  $f(x) \in C^\alpha(\bar{D})$ ,  $g(x) \in C^{2+\alpha}(\bar{D})$ ,  $g(x) \geq 0$  on  $\Gamma_1$  and  $g(x) = 0$  on  $\Gamma_2$ , and  $\partial D$  is in  $C^{2+\alpha}$ . Here  $C^{m+\alpha}(\bar{D})$  denotes the space of functions whose derivatives up to order  $m$  are Holder continuous with  $0 < \alpha < 1$ .

It is shown in [10] (Theorem 3.2, p 26) that there exists a unique solution  $u(x)$  for problem 2.1 and  $u \in W^{2,p}(D)$  for any  $p < \infty$ . The Sobolev imbedding theorem ([22], p 85) says immediately that the solution of problem 2.1 satisfies

$$u(x) \in C^{1,\lambda}(\bar{D}), \quad 0 < \lambda < 1, \tag{2.5}$$

therefore,  $u(x)$  has continuous first order derivatives in  $D$ .

To solve the free boundary Problem 2.1, we need to determine the free boundary  $\Gamma_{u_2} = \overline{\Omega_2} \cap \overline{D_2 - \Omega_2}$ , which is located inside the right domain  $D_2$ . The solution  $u(x)$  should satisfy the partial differential equation (2.1) in  $\Omega$  with boundary conditions (2.3) and  $u = 0$  on  $\Gamma_{u_2}$ ; while in  $D_2 - \Omega_2$ ,  $u(x) = 0$ . The free boundary Problem 2.1 is difficult to solve. In addition to determining the unknown  $u(x)$ , the free boundary also needs to be determined. Many industrial and engineering problems belong to this category.

Friedman [10] showed that the free boundary Problem 2.1 is equivalent to the following variational inequality, which can be handled conveniently both from theoretical and numerical perspectives.

**Problem 2.2.** Let  $H = \{u : u \in H^1(D), u = g(x) \text{ on } \partial D, u \geq 0 \text{ in } D\}$ . Find  $u \in H$ , such that

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in H, \tag{2.6}$$

where

$$a(u, v) = \int_D \left[ \sum_{i=1}^2 \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} + c(x)u(x)v(x) \right] dx,$$

$$\langle f, v \rangle = \int_D f(x)v(x)dx.$$

Since  $H$  is a closed convex subset of  $H^1(D)$ , it is well known [23] that there exists a unique solution  $u$  to (2.6). With  $u$  determined in  $D$ , we can define  $\Omega_2 = \{(x_1, x_2) \in D_2 : u(x_1, x_2) > 0\}$ . This also yields the solution  $u(x)$  of Problem 2.1.

Numerical schemes for variational inequalities have been investigated by many researchers. As to the free boundary Problem 2.1, the free boundary  $\Gamma_{u_2}$  lies in  $D_2$  only, therefore, we can consider the solution  $u$  of (2.1) to satisfy two related problems: One in  $D_1$  governed by a partial differential equation and one in  $D_2$  governed by a variational inequality. In  $D_1$ ,  $u_1 = u|_{D_1}$  satisfies a partial differential equation (2.1); while in  $D_2$ ,  $u_2 = u|_{D_2}$  satisfies a variational inequality. It is natural to construct a nonoverlapping domain decomposition method to solve for  $u$  in these two different regions, respectively. Currently there is no literature reporting on nonoverlapping domain decomposition methods for the general variational inequality problem. Some research work has been done for the case when  $D_1$  and  $D_2$  have a common region which is used to transfer data successively. This is a possible approach, but is not what we shall use here. Here we shall use the approach of splitting the original region into two nonoverlapping subdomains  $D_1$  and  $D_2$  which have the common boundary  $D_1 \cap D_2 = \Gamma_0$ , and then successively solving these two problems (which are coupled on  $\Gamma_0$ ) in  $D_1$  and  $D_2$ , respectively.

We shall construct a new nonoverlapping domain decomposition method in Algorithm 2.1 based on the above idea and provide the convergence analysis in Section 3.

**Lemma 2.1.** *Define*

$$H_1 = \{w(x) : w \in H^1(D_1), w|_{\Gamma_1} = g(x)\}$$

and

$$H_2 = \{w(x) : w \in H^1(D_2), w \geq 0 \text{ in } D_2, w|_{\Gamma_2} = 0\}.$$

Let  $u(x)$  be the solution of Problem 2.1.  $u_1 \in H_1$  and  $u_2 \in H_2$  be the restriction of  $u$  in  $D_1$ ,  $D_2$ , respectively, i.e.,  $u_1 = u|_{D_1}$  and  $u_2 = u|_{D_2}$ . Then  $u_1$  and  $u_2$  will satisfy

$$a_1(u_1, v) = \langle f, v \rangle_1 + \int_{\Gamma_0} \frac{\partial u_1}{\partial n_1} v ds \quad \forall v \in H^1(D_1), \quad v|_{\Gamma_1} = 0 \quad (2.7)$$

$$a_2(u_2, v - u_2) \geq \langle f, v - u_2 \rangle_2 + \int_{\Gamma_0} \frac{\partial u_2}{\partial n_2} (v - u_2) ds \quad \forall v \in H_2, \quad (2.8)$$

where

$$a_i(u, v) = \int_{D_i} \left[ \sum_{j=1}^2 \frac{\partial u(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} + c(x)u(x)v(x) \right] dx,$$

$$\langle f, v \rangle_i = \int_{D_i} f(x)v(x)dx,$$

$i = 1, 2$ .  $n_1, n_2$  is the normal direction on  $\Gamma_0$  from  $D_1, D_2$ , respectively.

**Proof.**  $u_1$  satisfies (2.1) in  $D_1$  and  $u_1 \in H_1$ . Let  $v \in H^1(D_1)$  and  $v|_{\Gamma_1} = 0$ . Then multiply (2.1) by  $v$  and use integration by parts in  $D_1$ . We have (2.7) immediately.

To prove (2.8), for any  $v \in H_2$ , which is well defined in  $D_2$  only, we need to extend  $v$  into  $D_1$  properly. To this end, let  $v|_{D_1}$  satisfy  $-\Delta v + c(x)v = 0$  in  $D_1$ ,  $v = g(x)$  on  $\Gamma_1$  and equal to the value of  $v$  on  $\Gamma_0$ . It is easy to see that this is a well-defined extension of  $v$  to the whole domain and the new  $v$  will be in  $H$ .

Since  $u$  is also a solution to Problem 2.2, we can deduce from (2.6)

$$a_1(u_1, v - u_1) + a_2(u_2, v - u_2) \geq \langle f, v - u_1 \rangle_1 + \langle f, v - u_2 \rangle_2. \quad (2.9)$$

Since  $v - u_1 \in H^1(D_1)$  and  $(v - u_1)|_{\Gamma_1} = 0$ , from (2.7)

$$a_1(u_1, v - u_1) = \langle f, v - u_1 \rangle_1 + \int_{\Gamma_0} \frac{\partial u_1}{\partial n_1} (v - u_1) ds.$$

Therefore, (2.9) becomes

$$\langle f, v - u_1 \rangle_1 + \int_{\Gamma_0} \frac{\partial u_1}{\partial n_1} (v - u_1) ds + a_2(u_2, v - u_2) \geq \langle f, v - u_1 \rangle_1 + \langle f, v - u_2 \rangle_2,$$

which yields (2.8) directly from the fact that  $\partial u_1/\partial n_1 = -\partial u_2/\partial n_2$  and  $u_1 = u_2$  on  $\Gamma_0$ , since  $u$  has continuous first-order derivatives in  $D$  from (2.5). ■

From Lemma 2.1, we found that the solution  $u$  satisfies a partial differential equation in  $D_1$  and satisfies a variational inequality in  $D_2$ . We propose a successive non-overlapping domain decomposition algorithm based on that observation, as follows:

**Algorithm 2.1.** Given  $g_1^0, g_2^0$  on  $\Gamma_0$ . Let  $n = 0$ .

Step 1. Solve the following two subproblems simultaneously:

**Subproblem 2.3.1.** Find  $u_1^n \in H_1$ , such that

$$a_1(u_1^n, v) + \int_{\Gamma_0} u_1^n v ds = \langle f, v \rangle_1 + \int_{\Gamma_0} g_1^n v ds \quad \forall v \in H^1(D_1), \quad v|_{\Gamma_1} = 0. \quad (2.10)$$

## 6 JIANG, BRUCH, AND SLOSS

**Subproblem 2.3.2.** Find  $u_2^n \in H_2$ , such that

$$a_2(u_2^n, v - u_2^n) + \int_{\Gamma_0} u_2^n(v - u_2^n)ds \geq \langle f, v - u_2^n \rangle_2 + \int_{\Gamma_0} g_2^n(v - u_2^n)ds \quad \forall v \in H_2. \quad (2.11)$$

Step 2. Define

$$g_2^{n+1} = 2u_1^n - g_1^n, \quad g_1^{n+1} = 2u_2^n - g_2^n. \quad (2.12)$$

Then repeat Step 1 with  $n$  replaced by  $n + 1$ .

**Proposition 2.1.** *By comparing (2.10) and (2.7), we can see that (2.10) is constructed naturally by utilizing the Robin boundary condition  $\partial u_1^n / \partial n_1 + u_1^n = g_1^n$  on  $\Gamma_0$ , because (2.7) is equivalent to*

$$a_1(u_1, v) + \int_{\Gamma_0} u_1 v ds = \langle f, v \rangle_1 + \int_{\Gamma_0} \left( \frac{\partial u_1}{\partial n_1} + u_1 \right) v ds \quad \forall v \in H^1(D_1), \quad v|_{\Gamma_1} = 0.$$

Then, (2.10) satisfied by  $u_1^n$  has the same format as the last equality for  $u_1$ . Therefore,  $u_1^n$  satisfies (2.1) with Robin boundary condition  $\partial u_1^n / \partial n_1 + u_1^n = g_1^n$  and  $u_1^n|_{\Gamma_0} = g_1^n$ , similarly as  $u_1$  does. This will lead us to solve (2.10) via the traditional finite difference method or finite element method.

**Proposition 2.2.** *Similarly, (2.11) and (2.8) have the same structure if we rewrite (2.8) as*

$$a_2(u_2, v - u_2) + \int_{\Gamma_0} u_2(v - u_2)ds \geq \langle f, v - u_2 \rangle_2 + \int_{\Gamma_0} \left( \frac{\partial u_2}{\partial n_2} + u_2 \right) (v - u_2) ds \quad \forall v \in H_2.$$

Therefore,  $u_2^n$  will satisfy (2.1) in the region where  $u_2^n > 0$  with Robin boundary condition  $\partial u_2^n / \partial n_2 + u_2^n = g_2^n$  as  $u_2$  does. We can then apply all the traditional numerical methods in the field of variational inequality to solve (2.11).

By combining the above two propositions, Subproblems 2.3.1 and 2.3.2 are none other than solving (2.1) in two different regions with Robin boundary conditions on the common boundary. Therefore, Algorithm 2.1 is a general algorithm that can be applied to all free boundary problems.

In Section 3, we shall show that  $\{u_1^n\}, \{u_2^n\}$  generated by Algorithm 2.1 converge to the solution of Problem 2.2, i.e.,  $u_1, u_2$ , in  $D_1, D_2$ , respectively, when  $n \rightarrow \infty$ . In Algorithm 2.1, two subproblems are solved independently by using the Robin boundary condition  $\partial u_i^n / \partial n_i + u_i^n = g_i^n$  on  $\Gamma_0$  in each iteration. Then, we update  $g_i^{n+1}$  with  $g_i^n$  for the next iteration. Therefore, Problem 2.2 can be solved in the two subdomains with the traditional nonoverlapping method. In Section 4, we will consider a free surface seepage problem as a model. In this example, we split the flow field into two nonoverlapping subdomains and apply Algorithm 2.1 there. The numerical result shows the efficiency and stability of Algorithm 2.1.

### III. CONVERGENCE ANALYSIS OF NONOVERLAPPING DOMAIN DECOMPOSITION METHODS

Define  $e_i^n = u_i^n - u_i$  in  $D_i$ ,  $\tilde{g}_i^n = g_i^n - [u_i + (\partial u_i / \partial n_i)]$  on  $\Gamma_0$ .

From (2.12) and the fact that  $u_1 = u_2$  and  $\partial u_2 / \partial n_2 = -\partial u_1 / \partial n_1$  on  $\Gamma_0$ , we have

$$\begin{aligned} \tilde{g}_2^{n+1} &= g_2^{n+1} - \left( \frac{\partial u_2}{\partial n_2} + u_2 \right) = 2u_1^n - g_1^n - \frac{\partial u_2}{\partial n_2} - u_2 = 2u_1^n - 2u_1 - g_1^n + \frac{\partial u_1}{\partial n_1} + u_1 \\ &= 2(u_1^n - u_1) - \left[ g_1^n - \left( \frac{\partial u_1}{\partial n_1} + u_1 \right) \right] = 2e_1^n - \tilde{g}_1^n \quad \text{on } \Gamma_0. \end{aligned} \quad (3.1)$$

Similarly,

$$\tilde{g}_1^{n+1} = 2e_2^n - \tilde{g}_2^n \quad \text{on } \Gamma_0. \quad (3.2)$$

Therefore,

$$\begin{aligned} \|\tilde{g}^{n+1}\|^2 &= \|\tilde{g}_1^{n+1}\|^2 + \|\tilde{g}_2^{n+1}\|^2 = \int_{\Gamma_0} (\tilde{g}_1^{n+1})^2 ds + \int_{\Gamma_0} (\tilde{g}_2^{n+1})^2 ds = \int_{\Gamma_0} (2e_2^n - \tilde{g}_2^n)^2 ds \\ &\quad + \int_{\Gamma_0} (2e_1^n - \tilde{g}_1^n)^2 ds = \|\tilde{g}^n\|^2 + 4 \int_{\Gamma_0} (e_2^n - \tilde{g}_2^n) e_2^n ds + 4 \int_{\Gamma_0} (e_1^n - \tilde{g}_1^n) e_1^n ds, \end{aligned}$$

where  $\|u\|$  is defined as the  $L^2$ -norm of function  $u \in L^2(\Gamma_0)$  on  $\Gamma_0$ ,  $\|\tilde{g}^n\|$  is regarded as a measurement for the norms of the errors  $\tilde{g}_1^n$  and  $\tilde{g}_2^n$  on  $\Gamma_0$ .

Rewrite (2.7) in the form of (2.10) as

$$a_1(u, v) + \int_{\Gamma_0} u_1 v ds = \langle f, v \rangle_1 + \int_{\Gamma_0} \left( u_1 + \frac{\partial u_1}{\partial n_1} \right) v ds \quad \forall v \in H^1(D_1), \quad v|_{\Gamma_1} = 0. \quad (3.4)$$

Subtraction of (3.4) from (2.10) gives

$$a_1(e_1^n, v) + \int_{\Gamma_0} e_1^n v ds = \int_{\Gamma_0} \tilde{g}_1^n v ds \quad \forall v \in H^1(D_1), \quad v|_{\Gamma_1} = 0. \quad (3.5)$$

Since  $e_1^n = u_1^n - u_1 \in H^1(D_1)$  and  $e_1^n|_{\Gamma_1} = 0$ , we replace  $v$  by  $e_1^n$  in (3.5) and have

$$\int_{\Gamma_0} (e_1^n - \tilde{g}_1^n) e_1^n ds = -a_1(e_1^n, e_1^n). \quad (3.6)$$

Therefore, we can replace the third term of the right hand side of (3.3) by  $-a_1(e_1^n, e_1^n)$  and obtain

$$\| \tilde{g}^{n+1} \|^2 = \| \tilde{g}^n \|^2 + 4 \int_{\Gamma_0} (e_2^n - \tilde{g}_2^n) e_2^n ds - 4a_1(e_1^n, e_1^n). \quad (3.7)$$

To simplify the second term on the right hand side of (3.7), we introduce the following lemma which will be proved at the end of this Section.

**Lemma 3.1.** *Sub-problem 2.3.2 is equivalent to the following problem:*

**Sub-problem 3.1.1.** *Find  $u_2^n \in H_2$  such that*

$$\begin{aligned} a_2(u_2^n, v - u_2^n) + \int_{\Gamma_0} u_2^n (v - u_2^n) ds &\geq a_2(u_2, v - u_2^n) - \int_{\Gamma_0} \frac{\partial u_2}{\partial n_2} (v - u_2^n) ds \\ &+ \int_{\Gamma_0} g_2^n (v - u_2^n) ds \quad \forall v \in H_2. \end{aligned} \quad (3.8)$$

Suppose Lemma 3.1 holds. (3.8) can be rewritten as

$$\begin{aligned} a_2(u_2^n - u_2, v - u_2^n) + \int_{\Gamma_0} (u_2^n - u_2) (v - u_2^n) ds &\geq \int_{\Gamma_0} \left[ g_2^n - \left( u_2 + \frac{\partial u_2}{\partial n_2} \right) \right] \\ &\times (v - u_2^n) ds \quad \forall v \in H_2. \end{aligned} \quad (3.9)$$

Letting  $v = u_2$  in the above inequality, we obtain

$$-a_2(e_2^n, e_2^n) - \int_{\Gamma_0} e_2^n e_2^n ds \geq - \int_{\Gamma_0} \tilde{g}_2^n e_2^n ds,$$

i.e.,

$$\int_{\Gamma_0} (e_2^n - \tilde{g}_2^n) e_2^n ds \leq -a_2(e_2^n, e_2^n).$$

Combining this inequality with (3.7) yields

$$\| \tilde{g}^{n+1} \|^2 \leq \| \tilde{g}^n \|^2 - 4a_1(e_1^n, e_1^n) - 4a_2(e_2^n, e_2^n). \quad (3.10)$$

Since  $a_1(e_1^n, e_1^n) \geq 0$ ,  $a_2(e_2^n, e_2^n) \geq 0$ , then  $\| \tilde{g}^{n+1} \|^2 \leq \| \tilde{g}^n \|^2$ . Therefore,  $\{ \| \tilde{g}^n \|^2 \}$  is a bounded sequence between 0 and  $\| \tilde{g}^1 \|^2 = M$ .

From (3.10),

$$a_1(e_1^n, e_1^n) + a_2(e_2^n, e_2^n) \leq \frac{1}{4} (\| \tilde{g}^n \|^2 - \| \tilde{g}^{n+1} \|^2). \quad (3.11)$$

Summation of (3.11) from  $n = 1$  to  $N$  yields

$$\sum_{n=1}^N [a_1(e_1^n, e_1^n) + a_2(e_2^n, e_2^n)] \leq \frac{1}{4} (\|\tilde{g}^1\|^2 - \|\tilde{g}^N\|^2) \leq \frac{1}{4} M.$$

This holds for arbitrarily large  $N$ . Therefore,

$$\lim_{n \rightarrow \infty} [a_1(e_1^n, e_1^n) + a_2(e_2^n, e_2^n)] = 0.$$

Since both terms in the above limit are non-negative, we have

$$\lim_{n \rightarrow \infty} a_1(e_1^n, e_1^n) = 0 \tag{3.12}$$

and

$$\lim_{n \rightarrow \infty} a_2(e_2^n, e_2^n) = 0. \tag{3.13}$$

To prove  $\|e_1^n\| \rightarrow 0$ , we should consider the following two cases:

**Case 1.**  $c(x) \geq C_0 > 0$ , then

$$a_1(e_1^n, e_1^n) = \int_{\Omega_1} \nabla e_1^n \nabla e_1^n dx + C_0 \int_{\Omega_1} e_1^n e_1^n dx \geq \min(1, C_0) \|e_1^n\|_{1, D_1}^2$$

Then (3.12) yields directly  $\|e_1^n\|_{1, D_1} \rightarrow 0$ . Similarly,  $\|e_2^n\|_{1, D_2} \rightarrow 0$ . This completes the proof of convergence of Algorithm 2.1.

**Case 2.**  $c(x) \geq 0$ . Then

$$a_1(e_1^n, e_1^n) \geq |e_1^n|_{1, D_1}^2.$$

(3.12) yields only  $|e_1^n|_{1, D_1} \rightarrow 0$ . Similarly,  $|e_2^n|_{1, D_2} \rightarrow 0$ . However,  $e_1^n|_{\Gamma_1} = 0$  in  $D_1$ , and application of Friedrich inequality will yield  $\|e_1^n\|_{1, D_1} \rightarrow 0$ . Similarly,  $\|e_2^n\|_{1, D_2} \rightarrow 0$ . This also makes Algorithm 2.1 convergent. Combining the above two cases, we obtain the main result in this article as Theorem 3.2.

**Theorem 3.2.** *Suppose  $\{u_1^n\}, \{u_2^n\}$  are obtained from Algorithm 2.1, then*

$$\lim_{n \rightarrow \infty} \|u_1^n - u_1\|_{1, D_1} = \lim_{n \rightarrow \infty} \|u_2^n - u_2\|_{1, D_2} = 0. \tag{3.14}$$

Finally, we will prove Lemma 3.1, which is the foundation of Theorem 3.2.

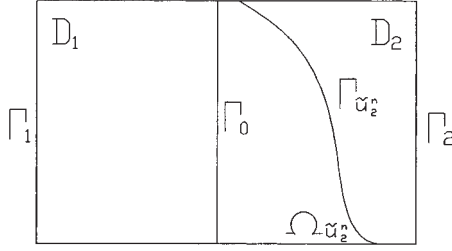


FIG. 2. The free boundary from approximate solution.

**Proof of Lemma 3.1.** Suppose  $\tilde{u}_2^n$  is the solution to Subproblem 3.1.1. We shall show that  $\tilde{u}_2^n$  is also the solution to Subproblem 2.3.2 (see Fig. 2).

From (3.8),

$$a_2(\tilde{u}_2^n, v - \tilde{u}_2^n) + \int_{\Gamma_0} \tilde{u}_2^n (v - \tilde{u}_2^n) ds \geq a_2(u_2, v - \tilde{u}_2^n) - \int_{\Gamma_0} \frac{\partial u_2}{\partial n_2} (v - \tilde{u}_2^n) ds + \int_{\Gamma_0} g_2^n (v - \tilde{u}_2^n) ds \quad \forall v \in H_2. \quad (3.15)$$

Define  $\Omega_{\tilde{u}_2^n} = \{(x_1, x_2) \in D_2 : \tilde{u}_2^n(x_1, x_2) > 0\}$ ,  $\Gamma_{\tilde{u}_2^n} = \overline{\Omega_{\tilde{u}_2^n}} \cap \overline{D_2} - \overline{\Omega_{\tilde{u}_2^n}}$ . It is obvious that  $\tilde{u}_2^n$  is positive inside  $\Omega_{\tilde{u}_2^n}$  and becomes 0 outside  $\Omega_{\tilde{u}_2^n}$ . ( $\Gamma_{\tilde{u}_2^n}$  is the free boundary of  $\tilde{u}_2^n$ ). Letting  $\tilde{e}_2^n = \tilde{u}_2^n - u_2$ , (3.15) can be rewritten as

$$a_2(\tilde{e}_2^n, v - \tilde{u}_2^n) + \int_{\Gamma_0} \tilde{e}_2^n (v - \tilde{u}_2^n) ds \geq \int_{\Gamma_0} \left[ g_2^n - \left( u_2 + \frac{\partial u_2}{\partial n_2} \right) \right] (v - u_2^n) ds. \quad (3.16)$$

For any  $w \in H_2$  satisfying  $w \geq 0$  in  $\Omega_{\tilde{u}_2^n}$ ,  $w = 0$  in  $D_2 - \Omega_{\tilde{u}_2^n}$ , let  $v_+ = \tilde{u}_2^n + \epsilon w$ ,  $v_- = \tilde{u}_2^n - \epsilon w$ , where  $\epsilon$  is small enough to make sure  $\tilde{u}_2^n + \epsilon w \geq 0$  and  $\tilde{u}_2^n - \epsilon w \geq 0$  in  $\Omega_{\tilde{u}_2^n}$ . Then  $v_+$  and  $v_-$  are both  $\geq 0$  in  $\Omega_{\tilde{u}_2^n}$  and  $= 0$  in  $D_2 - \Omega_{\tilde{u}_2^n}$ .

Replacing  $v$  by  $v_+$  and  $v_-$ , respectively, we obtain the following equality:

$$a_2(\tilde{e}_2^n, w) + \int_{\Gamma_0} \tilde{e}_2^n w ds = \int_{\Gamma_0} \left[ g_2^n - \left( u_2 + \frac{\partial u_2}{\partial n} \right) \right] w ds, \quad (3.17)$$

for any  $w \in H_{\tilde{u}_2^n} = \{u \in H_2 : u \geq 0 \text{ in } \Omega_{\tilde{u}_2^n}, u = 0 \text{ in } D_2 - \Omega_{\tilde{u}_2^n}\}$ . Under integration by parts on  $\Omega_{\tilde{u}_2^n}$  to the term on the left-hand side, (3.17) becomes

$$\int_{\Gamma_0} \left( \frac{\partial \tilde{e}_2^n}{\partial n} + \tilde{e}_2^n \right) w ds + \int_{\Omega_{\tilde{u}_2^n}} (-\Delta \tilde{e}_2^n + c \tilde{e}_2^n) w dx = \int_{\Gamma_0} \left[ g_2^n - \left( u_2 + \frac{\partial u_2}{\partial n} \right) \right] w ds. \quad (3.18)$$

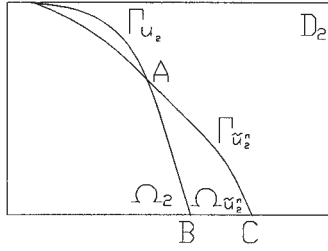


FIG. 3. Case which does not exist.

By choosing  $w \in H_{\tilde{u}_2^n}$  such that  $w|_{\Gamma_0} = 0$ , we have

$$\int_{\Omega_{\tilde{u}_2^n}} (-\Delta \tilde{e}_2^n + c \tilde{e}_2^n) w dx = 0.$$

This yields directly from the arbitrariness of  $w$  that

$$-\Delta \tilde{e}_2^n + c \tilde{e}_2^n = 0 \quad \text{in } \Omega_{\tilde{u}_2^n},$$

i.e.,

$$-\Delta \tilde{u}_2^n + c \tilde{u}_2^n = -\Delta u + cu \quad \text{in } \Omega_{\tilde{u}_2^n}. \tag{3.19}$$

Then, (3.18) becomes

$$\int_{\Gamma_0} \left( \frac{\partial \tilde{e}_2^n}{\partial n} + \tilde{e}_2^n \right) w ds = \int_{\Gamma_0} \left[ g_2^n - \left( u_2 + \frac{\partial u_2}{\partial n} \right) \right] w ds.$$

From the definition of  $\tilde{e}_2^n$ , this can be simplified as

$$\int_{\Gamma_0} \left( \frac{\partial \tilde{u}_2^n}{\partial n} + \tilde{u}_2^n \right) w ds = \int_{\Gamma_0} g_2^n w ds,$$

and because of the arbitrariness of  $w$  on  $\Gamma_0$ , we obtain

$$\frac{\partial \tilde{u}_2^n}{\partial n} + \tilde{u}_2^n = g_2^n \quad \text{on } \Gamma_0. \tag{3.20}$$

Next we will show that  $\Omega_2 \supset \Omega_{\tilde{u}_2^n}$ , i.e.,  $\Gamma_{\tilde{u}_2^n}$  is located inside  $\Omega_2$ . Assume it is wrong, then part of  $\Omega_{\tilde{u}_2^n}$  will be located outside  $\Omega_2$ , shown as the region  $D_{ABC}$  enclosed by  $A, B, C$  as in Fig. 3.  $\tilde{e}_2^n$  satisfies the following equation in  $D_{ABC}$ :

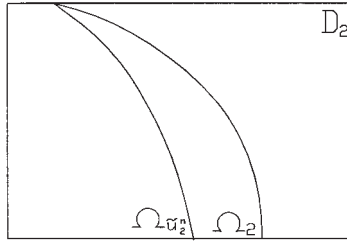


FIG. 4. Free boundary of the solution and the approximate solution.

$$-\Delta \tilde{e}_2^n + c\tilde{e}_2^n = 0 \quad \text{in } D_{ABC}. \quad (3.21)$$

Because  $\tilde{u}_2^n > 0$  and  $u_2 = 0$  in  $D_{ABC}$ , we have  $\tilde{e}_2^n > 0$  in  $D_{ABC}$ . Meanwhile,  $\tilde{e}_2^n|_{AB} = \tilde{u}_2^n - u_2 = \tilde{u}_2^n \geq 0$ ;  $\tilde{e}_2^n|_{BC} = \tilde{u}_2^n - u_2 = \tilde{u}_2^n \geq 0$ ;  $\tilde{e}_2^n|_{AC} = \tilde{u}_2^n - u_2 = 0 - 0 = 0$ . Therefore,  $\min \tilde{e}_2^n|_{D_{ABC}} = 0 = \tilde{e}_2^n|_{AC}$ , i.e.,  $\tilde{e}_2^n$  can obtain the minimum value on  $AC$ .

From the strong maximum principle [24], which is applied to (3.21), we have

$$\frac{\partial \tilde{e}_2^n}{\partial n} < 0 \quad \text{on } AC. \quad (3.22)$$

However, since  $u_2$  is the solution to the free boundary Problem 2.1 and  $AC$  is in the region where  $u_2 = 0$ , then  $\partial u_2 / \partial n|_{AC} = 0$ . Meanwhile  $\tilde{u}_2^n$  is also the solution to the free boundary Problem 3.1.1 and  $AC$  is part of the free boundary where both the solution and its normal derivative are zero, then  $\partial \tilde{u}_2^n / \partial n|_{AC} = 0$ . Therefore,

$$\frac{\partial \tilde{e}_2^n}{\partial n} = \frac{\partial \tilde{u}_2^n}{\partial n} - \frac{\partial u_2}{\partial n} = 0 \quad \text{on } AC,$$

which contradicts (3.22). Then our assumption on  $\Omega_2$  and  $\Omega_{\tilde{u}_2^n}$  is wrong. We conclude that  $\Omega_2 \supset \Omega_{\tilde{u}_2^n}$  holds as in Fig. 4.

Since  $u_2$  satisfies  $-\Delta u_2 + cu_2 = f$  in  $\Omega_2$ , then  $-\Delta u_2 + cu_2 = f$  in  $\Omega_{\tilde{u}_2^n}$ . Combining this and (3.19), we have

$$-\Delta \tilde{u}_2^n + c\tilde{u}_2^n = f \quad \text{in } \Omega_{\tilde{u}_2^n}. \quad (3.23)$$

In short,  $\tilde{u}_2^n$  satisfy (3.20) and (3.23), which is equivalent to (2.11) from Proposition 2.2.

Therefore,  $\tilde{u}_2^n$  is the solution of Subproblem 2.3.2. Since Subproblems 3.1.1 and 2.3.2 all have a unique solution, we can see that their solutions are the same based on the above conclusion.

This completes the proof of Lemma 3.1.  $\blacksquare$

#### IV. FREE SURFACE SEEPAGE EXAMPLE

As an example of the method, we consider the problem of free surface seepage (see Fig. 5): find the free surface in a steady, two-dimensional seepage through a rectangular dam.

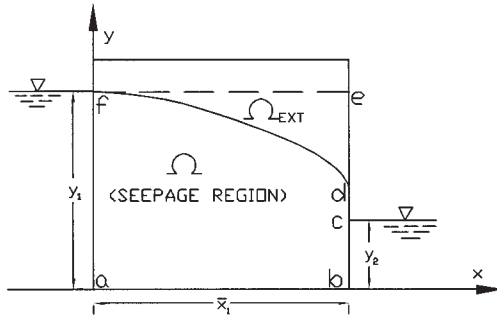


FIG. 5. Free boundary seepage problem.

In this study, the free surface, whose position is not known in advance, is to be found. In the seepage region  $\Omega$  with  $(x_1, x_2) = (x, y)$ , the velocity potential  $\phi$  must satisfy the following:

$$\begin{aligned}
 \Delta \phi &= 0 && \text{in } \Omega \\
 \phi &= y_1 && \text{on } [af] \\
 \phi &= y_2 && \text{on } [bc] \\
 \phi &= y && \text{on } [cd] \\
 \phi &= y && \text{on } \widehat{fd} \\
 \phi_\eta &= 0 && \text{on } \widehat{fd} \\
 \phi_\eta &\leq 0 && \text{on } [cd],
 \end{aligned} \tag{4.1}$$

where  $y_1$  and  $y_2$  is the height of the water on the left and right sides, respectively. The flow domain  $\Omega$  is not known since the location of the free surface  $\widehat{fd}$  needs to be found. Define  $D$  to be  $D = \{(x, y) : 0 < x < \bar{x}_1, 0 < y < y_1\}$  and define  $\bar{\phi}$  as an extension of  $\phi$  as follows:

$$\bar{\phi} = \begin{cases} \phi(x, y) & \text{in } \Omega \\ y & \text{in } \bar{D} - \Omega = \Omega_{ext}. \end{cases} \tag{4.2}$$

Using the Baiocchi transformation, a new variable is defined as

$$w(x, y) = \int_y^{y_1} [\bar{\phi}(x, \bar{\eta}) - \bar{\eta}] d\bar{\eta}. \tag{4.3}$$

Then  $w$  satisfies

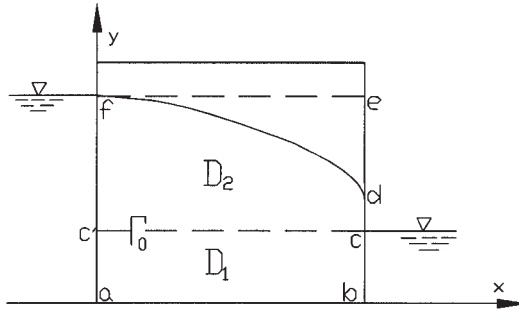


FIG. 6. Domain decomposition.

$$\begin{aligned}
 \Delta w &= \chi_{\Omega} & \text{in } D \\
 w(0, y) &= \frac{1}{2}(y_1 - y)^2 & \text{on } [af] \\
 w(x, 0) &= \frac{y_1^2}{2} - \frac{y_1^2 - y_2^2}{2\bar{x}_1}x & \text{on } [ab] \\
 w(\bar{x}_1, y) &= \frac{1}{2}(y_2 - y)^2 & \text{on } [bc] \\
 w &= 0 & \text{in } \bar{D} - \bar{\Omega}.
 \end{aligned} \tag{4.4}$$

If  $w$  is found satisfying (4.4), the following quantities can be obtained:

$$\begin{aligned}
 \Omega &= \{(x, y) : (x, y) \in D, w(x, y) > 0\} \\
 \text{graph } \bar{f} &= \partial\Omega - \partial D = \text{points of } \partial\Omega \text{ not in } \partial D \\
 \phi &= y - w_y & \text{in } \Omega.
 \end{aligned}$$

Next decompose  $D$  into two nonoverlapping regions  $D_1$  and  $D_2$  with common boundary  $\Gamma_0$  (See Fig. 6) such that  $D_2$  is a region containing the free surface. Consider the following coupled problems.

Assume  $g_1 = g_2 = 0$  on  $\Gamma_0$  initially.

Problem 1: Find  $w_1$  such that  $w_1 \in K_1 = \{v | v \in H_1(D_1), v > 0 \text{ in } D_1\}$ ,

$$\begin{aligned}
 \Delta w_1 &= f(x, y) & \text{in } D_1 \\
 \frac{\partial w_1}{\partial n} + w_1 &= g_1 & \text{on } \Gamma_0 \\
 w_1(0, y) &= \frac{1}{2}(y_1 - y)^2 & \text{on } [ac']
 \end{aligned}$$

$$\begin{aligned}
 w_1(x, 0) &= \frac{y_1^2}{2} - \frac{y_1^2 - y_2^2}{2\bar{x}_1} x && \text{on } [ab] \\
 w_1(\bar{x}_1, y) &= \frac{1}{2}(y_2 - y)^2 && \text{on } [bc]
 \end{aligned} \tag{4.5}$$

Problem 2: Find  $\{w_2, \Omega_2\}$  such that  $w_2 \in K_2 = \{v|v \in H_1(D_2), v \geq 0 \text{ in } D_2\}$ ,

$$\begin{aligned}
 \Delta w_2 &= f(x, y)\chi_{\Omega_2} && \text{in } D_2 \\
 \frac{\partial w_2}{\partial n} + w_2 &= g_2 && \text{on } \Gamma_0 \\
 w_2(0, y) &= \frac{1}{2}(y_1 - y)^2 && \text{on } [c'f] \\
 w_2(x, y) &= 0 && \text{on } [ce] \cup [ef]
 \end{aligned} \tag{4.6}$$

where  $\Omega_2 = \{(x, y)|w_2(x, y) > 0\}$ .

Problem 3: Let  $g_1 = 2w_2 - g_2, g_2 = 2w_1 - g_1$  on  $\Gamma_0$  and solve Problems 1 and 2 iteratively. From Propositions 2.1 and 2.2, we can see that (4.5) and (4.6) are actually the direct application of Algorithm 2.1 to the free surface seepage problem.

To solve (4.5) and (4.6) numerically, we will use the SOR (successive over-relation) method combined with projection. Bruch [18] showed that this scheme can be used effectively in the study of the seepage problem.

When applying SOR, (4.6) becomes:

$$w_{ij}^{[m+(1/2)]} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \left[ \frac{1}{\Delta x^2} (w_{i+1,j}^{(m)} + w_{i-1,j}^{(m+1)}) + \frac{1}{\Delta y^2} (w_{i,j+1}^{(m)} + w_{i,j-1}^{(m+1)}) - 1 \right] \tag{4.7}$$

and

$$w_{ij}^{(m+1)} = \max(0, w_{ij}^{(m)} + \bar{\alpha}(w_{ij}^{[m+(1/2)]} - w_{ij}^{(m)}),$$

where  $\bar{\alpha}$  is the relaxation parameter and  $i, j$  are the column and row mesh point numbers, respectively. Equation (4.5) can be handled similarly without the max operator.

Convergence on  $D_1$  and  $D_2$  is determined when

$$\max_{ij} |w_{1(i,j)}^{(n+1)} - w_{1(i,j)}^{(n)}| < \epsilon \quad \text{and} \quad \max_{ij} |w_{2(i,j)}^{(n+1)} - w_{2(i,j)}^{(n)}| < \epsilon,$$

respectively, where  $\epsilon$  is some fixed positive constant.

The numerical example problem investigated used the following data:  $y_1 = 1.00, y_2 = \frac{1}{6}, \bar{x}_1 = \frac{2}{3}, \bar{\alpha} = 1.85, \epsilon = 1 \times 10^{-5}, \Delta x = 0.0069, \Delta y = 0.01$ .

The free surface is taken as the first mesh point with a value of  $w$  that is less than  $\epsilon$  when you move in the vertical direction for a fixed  $x$ . The final free surface location based on (4.5) and

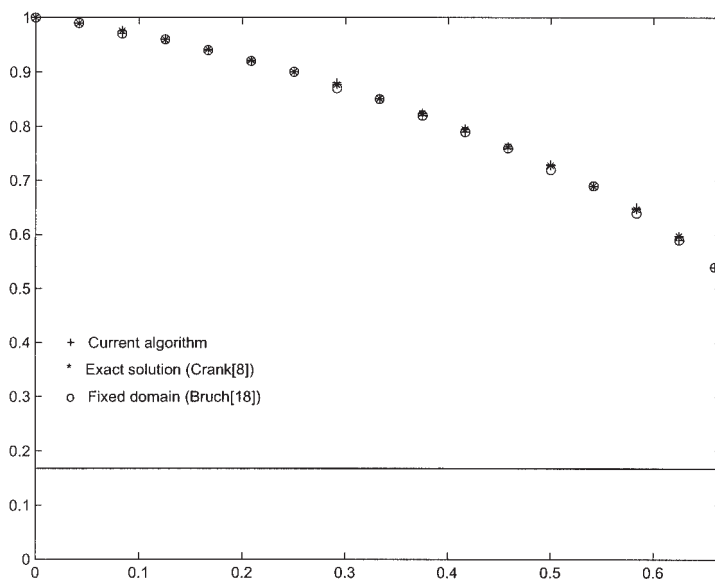


FIG. 7. Numerical result.

(4.6) is shown in Fig. 7 along with the numerical result by the fixed domain method as in [18] and the exact solution reported by Crank [8], attributed to Polubarinova-Kochina. We can see that our numerical method is as good as the traditional fixed domain method in approximating the exact solution of the seepage problem. The total number of iteration steps required to reach the tolerance is 461.

## V. CONCLUSION AND FUTURE DIRECTIONS

In this article, we studied the variational inequality arising from a free boundary problem. The characteristic of this problem is that the free boundary is unknown in advance. However, we can determine that the free boundary is located in one of the subdomains if we can properly split the domain into two or more subdomains. Then we can apply the traditional nonoverlapping domain decomposition method to this problem where in one subdomain, a partial differential equation is solved while in the other domain a variational inequality is considered. We have shown that this nonoverlapping DD method is convergent.

However, there are other kinds of variational inequalities whose free boundaries are absolutely unknown in advance, therefore the above technique does NOT apply. In our future work, we will construct a new algorithm for this general variational inequality problem and provide the convergence analysis there.

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