
Numerical Simulation of Free Seepage Flow on Non-matching Grids

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Summary. A new domain decomposition technique on non-matching grids for free boundary problems is considered. An iterative DD scheme is used to reduce the original free boundary problem to a sequence of problems on two subdomains, one of which includes the free boundary and is described by a variational inequality and the other includes the remainder of the problem and is described by a second order partial differential equation. In the subdomain which contains the free boundary, a fine grid is utilized in order to capture the free boundary more precisely; while in the other subdomain, a coarse grid is used in order to speed up the computation. At each step of the iteration, two sub-problems are solved by using a Robin boundary condition on the interface.

1 Introduction

Both overlapping and non-overlapping domain decomposition (DD) methods have been intensively studied for partial differential equations, see e.g. [9, 5, 10, 6, 2]. In the last few decades, mathematicians began to apply the overlapping domain decomposition methods to solve variational inequality problems. The basic idea is to split the original domain into several overlapping subdomains and solve the variational inequality on each subdomain via data transfer from the common area between those subdomains. [1, 8, 3] and their references provide many variants of this approach whereas convergence analyses of the algorithms and their application to many problems in different fields are provided.

However, for many practical problems in the engineering and industrial fields, it is much easier and more convenient to split the original domain into two or three non-overlapping subdomains and then take care of the problems in each subdomain where the original problem may show different behavior. Recently, a non-overlapping DD method which utilizes a Robin boundary condition on the common boundary

between these subdomains was proposed in [4] for the variational inequality problem and a convergence analysis of the DD method was provided.

In this paper, we consider free seepage flow through a dam with a toe drain that can be considered as a variational inequality. We will apply the non-overlapping DD method to decompose the original problem into two sub-problems where the partial differential equation is treated in one subdomain while the variational inequality is treated in the second subdomain where the free boundary is located. Since our concern is to find the exact location of the free boundary in the second domain, non-matching grids are applied in those subdomains. In the first subdomain, a coarse grid is used in order to speed up the computation; while in the second subdomain which contains the free boundary, a fine grid is utilized in order to capture the free boundary more precisely. At each step of the iteration, two sub-problems are solved simultaneously by using a Robin boundary condition on the common boundary.

This paper is organized as follows. In Section 2, we formulate the seepage problem and apply the non-overlapping DD method to split the original problem into 2 sub-problems. In Section 3, we utilize the non-matching grid technique in those two subdomains for the discretization of the problem and then apply a finite difference method with projection on the non-matching grids. Numerical results are reported to show the advantage of our new algorithm. In Section 4, a summary of the paper and some future considerations are outlined.

2 Formulation of the problem

Many problems involving free boundaries can be reduced to the study of variational inequalities. In [1], the author proposed several domain decomposition methods to split the domain into two or more subdomains. Then by iterating between these subdomains he solved the whole problem and found the free boundary. However, these schemes require the solution of one problem at a time. Herein, a DD scheme will be used simultaneously.

In this paper, we consider a free boundary seepage problem of flow through a porous dam with a toe drain. For simplicity, the soil in the flow field is assumed to be homogeneous and isotropic, capillary and evaporation effects are neglected. In addition, the flow follows Darcy’s law:

$$\vec{q} = -k\nabla[(\frac{p}{\rho g}) + y], \tag{2.1}$$

where \vec{q} is the velocity vector, p is the pressure, k is the permeability of the soil, ρ is the density of the fluid, g is the gravitational acceleration, and y is the vertical coordinate(positive upward). The seepage velocity has a potential $\phi(x, y) = k[(\frac{p}{\rho g}) + y]$. Meanwhile, let $\psi(x, y)$ be the stream function of the flow. In this study, the location of the free surface $\Gamma_0 = \{x, \bar{f}(x)\}$ and the seepage domain Ω need to be found, see Figure 1. The seepage domain is defined as:

$$\Omega = \{(x, y) : 0 < x \leq x_F, 0 < y < \alpha(x); x_F < x < x_C, 0 < y < \bar{f}(x)\},$$

where x_F and x_C are the distances in the x -direction to points F and C , respectively, and $\alpha(x)$ is the shape function of the dam profile.

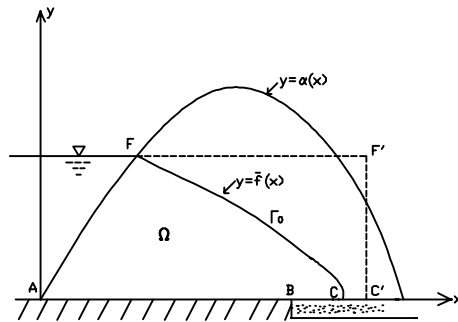


Fig. 1. The seepage problem.

The functions $\phi(x, y)$ and $\psi(x, y)$ are defined on $\overline{\Omega}$ and satisfy the following formulation:

$$\begin{aligned}
 \Omega &= \{(x, y) : 0 < x \leq x_F, 0 < y < \alpha(x); x_F < x < x_C, 0 < y < \overline{f}(x)\} \\
 \phi_x - \psi_y &= 0 && \text{in } \Omega \\
 \phi_y + \psi_x &= 0 && \text{in } \Omega \\
 \phi &= y_F && \text{on } \widehat{AF} \\
 \phi &= 0 && \text{on } [BC] \\
 \psi &= q && \text{on } [AB] \\
 \psi &= 0 && \text{on } \Gamma_0 \\
 \phi &= y && \text{on } \Gamma_0,
 \end{aligned} \tag{2.2}$$

where y_F is the height at F , and q is the flow rate through the flowfield.

Define $D = \{(x, y) : 0 < x \leq x_F, 0 < y < \alpha(x); x_F < x < x_{C'}, 0 \leq y < y_F\}$ and extend ϕ and ψ continuously to \overline{D} by setting $\overline{\phi}(x, y) = \phi(x, y)$ in $\overline{\Omega}$; y in $\overline{D} - \overline{\Omega}$ and $\overline{\psi}(x, y) = \psi(x, y)$ in $\overline{\Omega}$; $= 0$ in $\overline{D} - \overline{\Omega}$.

Next, we can define a new dependent variable w using the Baiocchi transformation on \overline{D} :

$$w(P) = \int_{\overline{FP}} -\overline{\psi}dx + (y - \overline{\phi})dy, \tag{2.3}$$

where \overline{FP} is a smooth path in D joining F to P in D . The integration is indeed independent of the path. Then $w \in H^2(D) \cap C^1(\overline{D})$ satisfies:

$$\begin{aligned}
 \Delta w &= \chi_\Omega && \text{in } D \\
 w_y &= y - y_F && \text{on } \widehat{AF} \\
 w &= \left(\frac{q^2}{6}\right) + q(x_B - x) && \text{on } [AB] \\
 w_y &= 0 && \text{on } [BC] \\
 w &= 0 && \text{in } \overline{D} - \overline{\Omega} \text{ (also on } \Gamma_0) \\
 w &> 0 && \text{in } \Omega \quad (w \geq 0 \text{ in } D),
 \end{aligned} \tag{2.4}$$

where $\chi_\Omega = 1$ in Ω and $\chi_\Omega = 0$ in $D - \Omega$. Hence,

$$w(x, y) \geq 0, \quad 1 - \Delta w(x, y) \geq 0, \quad w(1 - \Delta w) = 0 \text{ in } D. \tag{2.5}$$

If w is found satisfying (2.4), then we can determine $\Omega = \{(x, y) \in D : w(x, y) > 0\}$.

It will be seen shortly that if we can properly split D into two non-overlapping subdomains, the free boundary is only located in one subdomain, which makes the original problem simpler. Therefore, the DD method looks promising for this free boundary problem. [1] applied the non-overlapping D-N algorithm proposed in [6] to the above problem and the numerical results show that this D-N algorithm is better than the traditional one-domain finite difference scheme. However, no convergence property of the D-N algorithm has been proven.

Recently, a convergence analysis was provided for a non-overlapping DD method on uniform meshes with a Robin boundary condition applied to the general free boundary problem represented as a variational inequality [4]. In the following we use that DD scheme from [4] to solve the above seepage problem. First, decompose D into subsets $D_1 = \{(x, y) : 0 < x < x_F, 0 < y < \alpha(x)\}$ and $D_2 = \{(x, y) : x_F < x < x_{C'}, 0 < y < y_F\}$ with the interface between D_1 and D_2 denoted by $\Gamma = \{(x, y) : x = x_F, 0 < y < y_F\}$ in Figure 2. If w_1, w_2 denote the restriction of w in D_1 and D_2 , respectively, we can write down the following iterative procedure:

Step 1. Initially set $g_1^1 = g_2^1 = 0$ on Γ .

Step 2. Solve the following two sub-problems for w_1^n and $\{w_2^n, \Omega_2^n\}$, $n = 1, 2, \dots$, respectively:

Problem 1:

$$\begin{aligned} \Delta w_1^n &= 1 && \text{in } D_1 \\ w_1^n &= \left(\frac{q^2}{6}\right) + q(x_B - x) && \text{on } [AF_1'] \\ (w_1^n)_y &= y - y_F && \text{on } \widehat{AF} \\ w_1^n + \frac{\partial w_1^n}{\partial n} &= g_1^n && \text{on } \Gamma. \end{aligned} \tag{2.6}$$

Problem 2:

$$\begin{aligned} w_2^n (\Delta w_2^n - 1) &= 0 && \text{in } D_2 \\ w_2^n &= \left(\frac{q^2}{6}\right) + q(x_B - x) && \text{on } [F_1' B] \\ w_2^n|_y &= 0 && \text{on } [BC'] \\ w_2^n &\geq 0 && \text{in } \overline{D_2} \\ w_2^n + \frac{\partial w_2^n}{\partial n} &= g_2^n && \text{on } \Gamma \\ \Omega_2^n &= \{(x, y) : w_2^n(x, y) > 0\}. \end{aligned} \tag{2.7}$$

Step 3. Set

$$\begin{aligned} g_2^{n+1} &= 2w_1^n - g_1^n && \text{on } \Gamma \\ g_1^{n+1} &= 2w_2^n - g_2^n && \text{on } \Gamma. \end{aligned} \tag{2.8}$$

Then repeat Step 2 with n replaced by $n + 1$. These iterations are stopped when $\max_{\Gamma} |w_1^{n+1} - w_1^n| < \varepsilon$ and $\max_{\Gamma} |w_2^{n+1} - w_2^n| < \varepsilon$, where ε is some fixed error tolerance. Problem 1 and 2 in Step 2 are solved using the Robin boundary condition values g_1^n and g_2^n which are updated iteratively from their previous value and the w value on the common boundary. This avoids the computation of the partial derivatives of w which may reduce the precision.

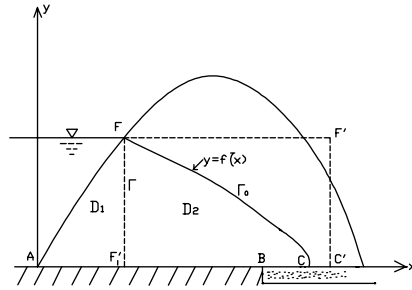


Fig. 2. The domain decomposition.

3 Non-Matching Grid Discretization and Results

In this Section, we will utilize a non-matching grid technique to obtain the numerical scheme for the above seepage problem on non-matching grids. At first, we apply the 2nd-order finite difference scheme to Δw and obtain the discrete formula for the first equation of (2.6) in D_1 as follows:

$$\frac{(w_1^n)_{i+1,j} + (w_1^n)_{i-1,j} + (w_1^n)_{i,j+1} + (w_1^n)_{i,j-1} - 4(w_1^n)_{i,j}}{h_1^2} = 1 \tag{3.1}$$

where D_1 is divided into a rectangular mesh with mesh size $\Delta x = \Delta y = h_1$, and where i, j are the row and column mesh point numbers, respectively. The boundary conditions in (2.6) can be discretized by a forward finite difference scheme.

Meanwhile, we can discretize (2.7) in D_2 in a similar way and obtain

$$(w_2^n)_{i,j} \left(\frac{(w_2^n)_{i+1,j} + (w_2^n)_{i-1,j} + (w_2^n)_{i,j+1} + (w_2^n)_{i,j-1} - 4(w_2^n)_{i,j}}{h_2^2} - 1 \right) = 0 \tag{3.2}$$

where D_2 is divided into a rectangular mesh with mesh size $\Delta x = \Delta y = h_2$.

Since our focus is to determine the location of the free boundary in D_2 more precisely, we construct a fine grid in D_2 and at the same time a coarse grid in D_1 to reduce the computation load there. Therefore, we assume $h_2 = \frac{1}{2}h_1$ throughout our computation, i.e., the grid size of D_2 is only half of that of D_1 . Because of the different grid sizes in D_1 and D_2 , the data transfer equations (2.8) between g_1^n and g_2^n cannot be discretized naturally. In order to discretize (2.8), we have to approximate g_1^n with its neighbouring g_2^n values on Γ , and vice versa, as shown in Figure 3. From Figure 3, we notice that $w_{1,j}$ depends on $w_{2,2j}$. Meanwhile, $w_{2,2j}$ is affected by $w_{1,j}$ and $w_{2,2j+1}$ is affected by both $w_{1,j}$ and $w_{1,j+1}$. Therefore, the reasonable data transfer on Γ will be $w_{1,j} = w_{2,2j}$ from D_2 to D_1 and $w_{2,2j} = w_{1,j}$ and $w_{2,2j+1} = \frac{1}{2}(w_{1,j} + w_{1,j+1})$ from D_1 to D_2 . g_1 and g_2 can be taken care of similarly. Then, (2.8) can be discretized as follows:

$$\begin{aligned} (g_1^{n+1})_j &= 2(w_2^n)_{2j} - (g_2^n)_{2j} \\ (g_2^{n+1})_{2j} &= 2(w_1^n)_j - (g_1^n)_j \\ (g_2^{n+1})_{2j+1} &= \frac{1}{2} \{ [2(w_1^n)_j - (g_1^n)_j] + [2(w_1^n)_{j+1} - (g_1^n)_{j+1}] \} \end{aligned} \tag{3.3}$$

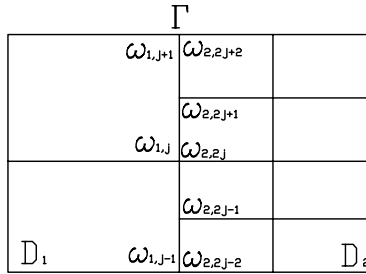


Fig. 3. The nonmatching grids.

The computation is run as follows: we compute w_1^n and w_2^n using (3.1) and (3.2). We then update g_1^n and g_2^n on Γ using (3.3) and repeat the scan through D_1 and D_2 to solve w_1^n and w_2^n . The iteration will stop if the convergence criterion is met.

During the computation, finite difference SOR (Successive over-relaxation) is utilized in D_1 , while in D_2 which contains the free boundary, finite difference SOR (Successive over-relaxation) with a projection is used to make sure the values of w_2 at each point are always non-negative.

Therefore, when applying the SOR in D_1 , (3.1) becomes:

$$\begin{aligned} (w_1^{(n+\frac{1}{2})})_{i,j} &= \frac{1}{4}((w_1^n)_{i+1,j} + (w_1^n)_{i-1,j} + (w_1^n)_{i,j+1} + (w_1^n)_{i,j-1} - h_1^2) \\ (w_1^{(n+1)})_{i,j} &= (w_1^n)_{i,j} + \bar{\beta}((w_1^{(n+\frac{1}{2})})_{i,j} - (w_1^n)_{i,j}) \end{aligned} \tag{3.4}$$

where $\bar{\beta}$ is the relaxation parameter.

Similarly, when applying the SOR with projection in D_2 , (3.2) becomes

$$\begin{aligned} (w_2^{(n+\frac{1}{2})})_{i,j} &= \frac{1}{4}((w_2^n)_{i+1,j} + (w_2^n)_{i-1,j} + (w_2^n)_{i,j+1} + (w_2^n)_{i,j-1} - h_2^2) \\ (w_2^{(n+1)})_{i,j} &= \max(0, (w_2^n)_{i,j} + \bar{\beta}((w_2^{(n+\frac{1}{2})})_{i,j} - (w_2^n)_{i,j})) \end{aligned} \tag{3.5}$$

The flow rate q through the flow field is also unknown *a priori*. Therefore, in addition to the inner iteration to solve for w with a given q , there is also an outer iteration on q to determine the flow rate. The compatibility condition for the outer iteration (see [7]) is $f(q) = (w_2(x_F, y_F - \Delta y)) - \frac{\Delta y^2}{2} = 0$. In fact, we can set q_0 and q_1 to arbitrary values. Then we use the secant method to determine q_2 based on q_0 and q_1 from (3.6) for the third outer loop, and so on until we reach some q_n whose $|f(q_n)| < \varepsilon$.

$$q_2 = q_1 - \frac{q_2 - q_1}{f(q_2) - f(q_1)} f(q_1). \tag{3.6}$$

The example uses the following data: $\alpha(x) = x$ where $0 < x < x_F$, $y_F = 30\text{ft}$, $x_F = 30\text{ft}$, $x_B = 60\text{ft}$, $h_1 = 0.5\text{ft}$, $h_2 = 0.25\text{ft}$, $\bar{\beta} = 1.25$, and $\varepsilon = 0.005$.

Figure 4 shows the free boundary obtained by the new DD algorithm. It exactly matches the numerical results from [1]. However, the combination of the new non-overlapping domain decomposition method and the non-matching grid technique

generates a better performance than that of [1]. Table 1 shows the required number of iterations for our current algorithm and the algorithm from [1]. We can see that the performance has been improved considerably over [1].

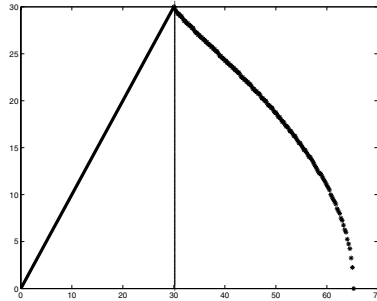


Fig. 4. Free boundary in D_2 .

Outer iteration	Current Algorithm	Algorithm of [1]
1	1867	2374
2	1024	1604
3	1651	2088
4	618	996
5	109	376
6	31	156
7	2	18

Table 1. Comparison of required number of inner iterations between the two algorithms.

4 Conclusion and future directions

In this paper, we studied a free boundary seepage problem of flow through a porous dam with a toe drain. The characteristic of this problem is that the free boundary is

unknown in advance. However, we can determine that the free boundary is located in one of the subdomains if we can properly split the domain into two or more subdomains. Then, we can apply the traditional non-overlapping DD method to this problem. Meanwhile, the non-matching grid discretization is utilized on these subdomains in order to obtain higher resolution of the free boundary in the fine-grid subdomain while maintaining computational efficiency in the coarse-grid subdomain.

The promising numerical results motivate us to establish the convergence analysis and error estimates between the numerical solution based on a combination of non-overlapping DD method and non-matching grid discretization and the true solution of the original problem for the general free boundary problem. We will investigate this theoretical issue in the future.

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