WILEY InterScience®
DISCOVER SOMETHING GREAT

A Priori Error Analysis of the Mortar Finite Element Method for Variational Inequalities

Bin Jiang
Department of Mathematics and Statistics, Portland State University, Portland, Oregon 97201

Received 26 July 2006; accepted 10 April 2007
Published online 19 June 2007 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/num.20279

The mortar finite element method is a special domain decomposition method, which can handle the situation where meshes on different subdomains need not align across the interface. In this article, we will apply the mortar element method to general variational inequalities of free boundary type, such as free seepage flow, which may show different behaviors in different regions. We prove that if the solution of the original variational inequality belongs to $H^2(D)$, then the mortar element solution can achieve the same order error estimate as the conforming P_1 finite element solution. Application of the mortar element method to a free surface seepage problem and an obstacle problem verifies not only its convergence property but also its great computational efficiency. Numer Methods Partial Differential Eq 24: 476–503, 2008

Keywords: mortar finite element method; domain decomposition; variational inequality; free seepage flow; obstacle problem

I. INTRODUCTION

Free boundary value problems have been an important topic in the field of computational science because many physical and engineering problems such as fluid flow in porous media, obstacle problems, elastic problems, and lubrication phenomena fall under this category. These problems can usually be transformed into variational inequalities. Reference 1 provided a detailed theoretical analysis about regularity of the solution and the free boundary. Meanwhile, different kinds of numerical methods are proposed to solve general free boundary problems, see [2–7] and the references therein. Among those methods, finite element method (FEM) is the most popular one. It is shown in [8, 9] that $\lim_{h\to 0} ||u_h - u|| = 0$ where u_h and u are the finite element solution and the original analytic solution, respectively. Therefore, the finite element solution can be considered as a good approximation to the analytic solution of the variational inequality when the mesh size h is small enough. Meanwhile, finite element method is also applied in solving application-oriented problems of free boundary type, such as Stefan or Hele–Shaw moving boundary problems, see [10, 11] for the error analysis.

Correspondence to: Bin Jiang, Department of Mathematics and Statistics, Portland State University, Portland, Oregon 97201 (e-mail: bjiang@pdx.edu)

In the last two decades, nonoverlapping domain decomposition (DD) finite element method has been successfully applied to solve linear partial differential equations, see [12–14] and the references therein. In [15, 16], we apply the nonoverlapping DD method to solve the variational inequality where a robin boundary condition is utilized on the common boundary between those subdomains. We show that at the continuous and discrete levels, the sequences of the approximate solutions based on domain decomposition converge to the original or the finite element solution, respectively.

The nonoverlapping DD methods proposed in [15, 16] assume the quasi-uniform meshes on the whole domain. However, the actual situation is perhaps very complicated and we may have different requirement in different parts of the domain. For example, in the subregion, which contains the free boundary, we may wish to apply the finer grid so as to catch the free boundary more precisely, while in the subdomain far away from the free boundary, the coarser grid would be a better choice. In [17], the nonmatching grid technique has been used to solve a free seepage flow problem so that better numerical performance has been observed than conforming methods on uniform meshes. Therefore, nonmatching grid technique, such as mortar element method, is well suited for our problem.

Mortar finite element method was introduced in [18, 19] in the last decade and since then it has been used extensively to solve linear partial differential equations (PDE). See [20–25] and the references therein. Meanwhile, it has also been applied to solve some kinds of variational inequalities, such as unilateral contact problems or Bingham fluids, where either the inequality of the problem is concentrated on the boundary (see [26]) or a nonlinear gradient term $|\nabla u|$ appears in the inequality (see [27]).

In this article, we shall apply mortar element method to variational inequalities of free boundary type. The major difference between PDE and variational inequality of free boundary type is that the solution of variational inequality is required to be nonnegative (or generally, not less than an obstacle function) while PDE has no such restriction. Therefore, we shall enforce nonnegativity for the function values in the domain. We will show that the mortar element method can achieve the same order error estimate as conforming P_1 element.

This article is organized as follows. In Section II, the variational inequality is formulated for general free boundary problem. The mortar finite element method is then applied to discretize the original domain into multiple subdomains and P_1 mortar element solution $\{u_h\}$ is formulated. In Section III, the error estimate between the mortar solution u_h and the analytic solution solution u is provided. In Section IV, the mortar element method is applied to solve a typical free seepage problem and an obstacle problem. Comparison with conforming finite element method shows its great computational efficiency. In Section V, conclusion of the article and some future efforts are given.

II. PROBLEM FORMULATION

Let us consider a general free boundary problem:

Find an open set $\Omega \subset D$ and $u(x) \in H^2(D)$ with $x = (x_1, x_2)$, such that

$$\begin{cases} (-\Delta u + c(x)u - f(x)) \cdot u(x) = 0 & \text{in } D, \\ u \geq 0 & \text{in } D, \\ -\Delta u + c(x)u - f(x) \geq 0 & \text{in } D, \\ u = g(x) & \text{on } \partial D, \\ \Omega = \{x \in D | u(x) > 0\}, \end{cases} \tag{2.1}$$

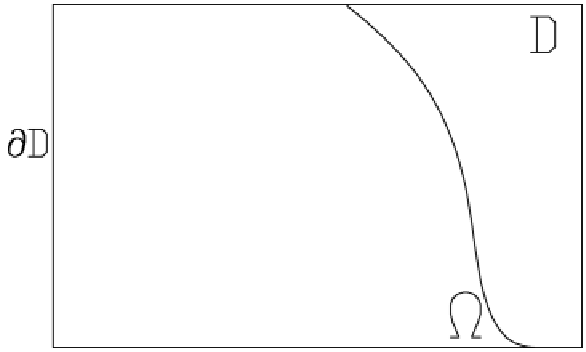

FIG. 1. The original free boundary problem.

where D is a bounded, polygonal 2D domain whose boundary is denoted by ∂D. $c(x) \geq 0$, $c(x) \in C^{\alpha}(\overline{D})$, $f(x) \in C^{\alpha}(\overline{D})$, $g(x) \in C^{2+\alpha}(\overline{D})$, $g(x) \geq 0$ on ∂D, and ∂D is in $C^{2+\alpha}$. Here $C^{m+\alpha}(\overline{D})$ denotes the space of functions whose derivatives up to order m are Holder continuous with $0 < \alpha < 1$ (Fig. 1). (2.1) also implies

$$-\Delta u + c(x)u = f(x) \quad \text{in } \Omega. \tag{2.2}$$

Friedman [1] showed that the free boundary Problem (2.1) is equivalent to the following variational inequality:

Find $u \in V = \{u : u \in H^1(D), \quad u = g(x) \text{ on } \partial D, \quad u \geq 0 \text{ in } D\}$, such that

$$a(u, v - u) \geq f(v - u) \quad \forall v \in V \tag{2.3}$$

where

$$a(u, v) = \int_D \left[\sum_{i=1}^{2} \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} + c(x)u(x)v(x) \right] dx,$$

$$f(v) = \int_D f(x)v(x)dx.$$

With u determined in D, we can define $\Omega = \{(x_1, x_2) \in D : u(x_1, x_2) > 0\}$.

Lemma 2.1 [1]. *Problem* (2.3) *has a unique solution* $u(x) \in W^{2,p}(D)$ *for any* $p < \infty$.

Since there is no general formula for the solution $u(x)$ of (2.1) or (2.3), different numerical methods have been developed for the approximate solution of (2.3). Since the major concern is to catch the free boundary, we shall apply the mortar element method to solve (2.3). That is, the finer grids will be constructed in the area close to the free boundary while the coarser grids will be constructed in the area far away from the free boundary.

To this end, let us decompose the domain D into several nonoverlapping subdomains D_k, $1 \leq k \leq N$,

$$\overline{D} = \bigcup_{k=1}^{N} \overline{D_k},$$

where each subdomain D_k has a regular quasi-uniform triangulation Σ_{h_k}, h_k is the maximum of the diameters of the triangles in Σ_{h_k} and $h = \max_{1\le k\le N} h_k$. For simplicity, we consider the geometrical conforming situation where the common boundary between any two different subdomains $\partial D_k \cap \partial D_l$, $k \neq l$ is either empty, or a vertex, or a common interface. The set of vertices of all triangles in D_k is denoted by N_k, while N_k^0 denotes those inside D_k. In each subdomain D_k, the conforming P_1 finite element space $P_1(D_k)$ is defined

$$P_1(D_k) = \{v : v|_\tau \in P_1(\tau) \quad \forall \tau \in \Sigma_{h_k}, \quad v \text{ is continuous at } p \in N_h,$$
$$v(p) = g(p) \text{ at } p \in N_h \cap \partial D\}.$$

Since Σ_{h_k} is both regular and quasi-uniform in D_k, there exists constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for any $\tau \in \Sigma_{h_k}$, $\frac{h_\tau}{\rho_\tau} \le \alpha_1$ where ρ_τ is the diameter of the largest inscribed circle inside τ, and $\frac{h_\tau}{h_k} \ge \alpha_2$. The inverse inequality holds within each D_k from the above assumption.

It is obvious that the restriction of Σ_{h_k} on the common interface of each pair of neighboring subdomains may not match. The mortar finite element method is characterized by introducing Lagrange multiplier space on these common interfaces. To this end, the common interface of the neighboring D_k and D_l is denoted by Γ_{kl} for D_k and Γ_{lk} for D_l, respectively. Meanwhile, Σ_{kl} denotes the restriction of Σ_{h_k} on Γ_{kl}, and Σ_{lk} the restriction of Σ_{h_l} on Γ_{lk}. N_{kl} and N_{lk} denote the set of nodes on Γ_{kl} and Γ_{lk}. It is obvious that Σ_{kl} and Σ_{lk} are different partitions for the common interface. We define a function as follows:

$$\delta(k,l) = \begin{cases} 1 & D_k \text{ and } D_l \text{ has a common interface} \\ 0 & \text{otherwise} \end{cases}$$

where $1 \le k, l \le N$, $k \neq l$. Therefore, $\delta(k,l) = \delta(l,k)$.

The skeleton composed of the common interfaces of all neighboring subdomains in D, $S = \bigcup_{k=1}^N \partial D_k \backslash \partial D$ can be decomposed as

$$S = \bigcup_{k=1}^{N} \bigcup_{\delta(l,k)=1} \Gamma_{kl},$$

where each interface appears twice in the above expression. For each common interface, we may select either Γ_{kl} or Γ_{lk} as the master side (called mortar) and the other as the slave side (called nonmortar). Then the union of either all mortar sides or all nonmortar sides will form a unique decomposition of the skeleton S. In this article, we assume the ratio of mesh sizes between any neighboring subdomains is uniformly bounded by a constant $C > 1$:

$$\text{If } \delta(k,l) = 1, h_k \le h_l, \text{ then } \frac{h_l}{h_k} \le C. \tag{2.4}$$

For each subdomain D_k, $1 \le k \le N$, we define $M(k) = \{1 \le l \le N : \Gamma_{kl} \text{ is a mortar}\}$ and $S(k) = \{1 \le l \le N : \Gamma_{kl} \text{ is a nonmortar}\}$, then $l \in M(k) \Longleftrightarrow k \in S(l)$. S will then have the following decomposition by all nonmortar sides:

$$S = \bigcup_{k=1}^{N} \bigcup_{l\in M(k)} \Gamma_{lk}.$$

Define the unconstrained product space as follows:

$$X_h = \prod_{k=1}^{N} P_1(D_k),$$

whose norms are defined as

$$\|v_h\|_{1,h}^2 = \sum_{k=1}^{N} \|v_h\|_{1,D_k}^2, \quad |v_h|_{1,h}^2 = \sum_{k=1}^{N} |v_h|_{1,D_k}^2, \quad \|v_h\|_{0,h}^2 = \sum_{k=1}^{N} \|v_h\|_{0,D_k}^2, \quad v_h \in X_h.$$

It is obvious that X_h is not a suitable discretization of V in (2.3) due to discontinuity across the interface and lack of nonnegativity on the whole domain. We will follow [18, 19] to construct a mortar element method where the jump of the function value is orthogonal to a modified function trace space on the common interfaces and the function is continuous across the vertices of all subdomains. To this end, let T denote the union of all vertices of D_k, $1 \le k \le N$. Meanwhile, We will also enforce nonnegativity for the function values at all active nodes, which include nodes from the mortar sides and nodes inside all subdomains, due to the fact that the function values on the remaining inactive nodes from the nonmortar sides can be uniquely determined from the corresponding mortar side for each common interface based on the jump orthogonality condition. In fact, Theorem 3.6 will show that the mortar element solution converges to u pointwisely on S. Therefore, the nonnegative constraint will also be satisfied on the nonmortar sides when h is small, see Remark 3.2.

The Lagrange multiplier space for the nonmortar sides is introduced as follows:

$$M_h = \prod_{k=1}^{N} \prod_{l \in M(k)} M_h(\Gamma_{lk}),$$

where

$$M_h(\Gamma_{lk}) = \{v \in C^0(\Gamma_{lk}) : v = w|_{\Gamma_{lk}}, w \in P_1(D_l),$$
$$v|_e = P_0(e), e \text{ is an edge containing an end point of } \Gamma_{lk}\}$$

is the modified restriction of $P_1(D_l)$ on Γ_{lk}.

Then the discretization space of V can be defined as

$$V_h = \Big\{ v \in X_h : b(v, \mu) = 0, \forall \mu \in M_h, v \text{ is continuous on } T,$$
$$v(P) \ge 0, P \in N_k^0 \text{ or } P \in \bigcup_{l \in M(k)} N_{kl}, 1 \le k \le N \Big\} \tag{2.5}$$

where

$$b(v, \mu) = \sum_{k=1}^{N} \sum_{l \in M(k)} \int_{\Gamma_{lk}} [v] \mu ds$$

and $[v] = v|_{D_l} - v|_{D_k}$.

The nonconforming mortar finite element formulation for (2.3) is given as follows: Find $u_h \in V_h$ such that

$$a(u_h, v_h - u_h) \geq f(v_h - u_h) \quad \forall v_h \in V_h, \tag{2.6}$$

where $a(u,v) = \sum_{k=1}^{N} \int_{D_k} [\nabla u \cdot \nabla v + cuv]dx$ and $f(v) = \int_D fvdx$.
The following Lemma shows the well-posedness of (2.6).

Lemma 2.2. *Problem (2.6) has a unique solution u_h.*

Proof. It is straightforward that $a(u,v)$ and $f(v)$ are continuous on V_h, which is a closed and convex subspace of X_h and $a(u,v)$ satisfies

$$a(u_h, v_h) \leq M\|u_h\|_{1,h} \cdot \|v_h\|_{1,h}, \quad \forall u_h, v_h \in V_h,$$

where $M > 0$ is a constant. Meanwhile, it can be shown similarly as in ([18], Sec.2.3) that $a(\cdot,\cdot)$ is V_h-elliptic, i.e.,

$$a(v_h, v_h) \geq \alpha\|v_h\|_{1,h}^2, \quad \forall v_h \in V_h.$$

Therefore, Lions–Stampacchia's theorem (see [28]) immediately implies the existence and uniqueness of the solution u_h from (2.6). ∎

Let I_k^h denote the Lagrange interpolation operator of order one on Σ_{h_k}. The following error estimate is given in [29]:

$$\left\|v - I_k^h v\right\|_{H^l(D_k)} \leq C h_k^{m-l} \|v\|_{H^m(D_k)}, \quad \forall v \in H^m(D_k) \tag{2.7}$$

where $0 \leq l \leq 1 \leq m \leq 2$. From now on, we assume C and $C_1, C_2, \ldots$ are generic constants independent of the mesh size h_k, $1 \leq k \leq N$.

The mortar projection is a basic tool for the analysis of the error estimate in [18, 19]. For each nonmortar side, the mortar projection is given by $\Pi_{lk} : H^1(\Gamma_{lk}) \to W_h(\Gamma_{lk})$,

$$\Pi_{lk}v(a) = v(a) \text{ and } \Pi_{lk}v(b) = v(b), \quad a, b \text{ are extremities of } \Gamma_{lk},$$
$$\int_{\Gamma_{lk}} (v - \Pi_{lk}v)\mu ds = 0, \quad \forall \mu \in M_h(\Gamma_{lk}), \tag{2.8}$$

where $W_h(\Gamma_{lk}) = \{v \in C^0(\Gamma_{lk}) : v = w|_{\Gamma_{lk}}, w \in P_1(D_l)\}$ is the restriction of $P_1(D_l)$ on Γ_{lk}.

It is shown in [18] that Π_{lk} is L^2, $H_{00}^{\frac{1}{2}}$ and H_0^1 stable: for $Y = L^2(\Gamma_{lk})$, $H_{00}^{\frac{1}{2}}(\Gamma_{lk})$ or $H_0^1(\Gamma_{lk})$,

$$\|\Pi_{lk}v\|_Y \leq C\|v\|_Y, \quad \forall v \in H_0^1(\Gamma_{lk}). \tag{2.9}$$

In Section III, we will show that the mortar element solution u_h of (2.6) can achieve the same error estimate as conforming P_1 finite element method, i.e., $u - u_h = O(h)$ where $h = \max_{1 \leq k \leq N} h_k$.

III. ERROR ESTIMATE

The following lemma is an extension of Falk's Lemma [29] to the nonoverlapping case.

Lemma 3.1. *Suppose $u \in V$ and $u_h \in V_h$ be solutions to* (2.3) *and* (2.6)*, respectively, and $u \in H^2(D)$, then*

$$\|u-u_h\|_{1,h}^2 \le C\left(\inf_{v_h\in V_h}\left\{\|u-v_h\|_{1,h}^2+\|u-v_h\|_{0,h}+\sum_{k=1}^N\sum_{l\in M(k)}\left|\int_{\Gamma_{lk}}\frac{\partial u}{\partial n_l}[v_h]ds\right| + \sum_{k=1}^N\left|\int_{\partial D_k\cap\partial D}\frac{\partial u}{\partial n_k}(u-v_h)ds\right|\right\} + \inf_{v\in V}\left\{\|v-u_h\|_{0,h}+\sum_{k=1}^N\left|\int_{\partial D_k\cap\partial D}\frac{\partial u}{\partial n_k}(v-u_h)ds\right|\right\} + \sum_{k=1}^N\sum_{l\in M(k)}\left|\int_{\Gamma_{lk}}\frac{\partial u}{\partial n_l}[u_h]ds\right|\right). \quad (3.1)$$

Proof.

$$\|u-u_h\|_{1,h}^2 \le 2\|u-v_h\|_{1,h}^2 + 2\|u_h-v_h\|_{1,h}^2 \quad \forall v_h\in V_h. \quad (3.2)$$

V_h-ellipticity of $a(\cdot,\cdot)$ from Lemma 2.2 implies

$$\begin{aligned}\alpha\|u_h-v_h\|_{1,h}^2 &\le a(u_h-v_h,u_h-v_h)\\ &= a(u_h-u,u_h-v_h)+a(u-v_h,u_h-v_h)\\ &= a(u_h,u_h-v_h)+a(u,v-u_h)-a(u,v)+a(u,v_h)\\ &\quad + a(u-v_h,u_h-v_h) \quad \forall v\in V. \end{aligned} \quad (3.3)$$

From (2.3), we have

$$-a(u,v) \le -a(u,u) - f(v-u) \quad \forall v\in V. \quad (3.4)$$

Similarly, (2.6) implies

$$a(u_h,u_h-v_h) \le f(u_h-v_h) \quad \forall v_h\in V_h. \quad (3.5)$$

Putting (3.4) and (3.5) into (3.3), we obtain

$$\begin{aligned}\|u_h-v_h\|_{1,h}^2 &\le \frac{1}{\alpha}\{f(u_h-v_h)+a(u,v-u_h)-a(u,u)-f(v-u)\\ &\quad + a(u,v_h)+a(u-v_h,u_h-v_h)\}\\ &= \frac{1}{\alpha}[a(u,v-u_h)-f(v-u_h)]+\frac{1}{\alpha}[a(u,v_h-u)-f(v_h-u)]\\ &\quad + \frac{1}{\alpha}a(u-v_h,u_h-v_h). \end{aligned} \quad (3.6)$$

By applying Green formula to the first pair of square brackets of (3.6), we have

$$
\begin{aligned}
a(u, v-u_h) - f(v-u_h) &= \sum_{k=1}^{N} \int_{D_k} \nabla u \cdot \nabla (v-u_h) + cu(v-u_h)dx - \int_D f(v-u_h)dx \\
&= \sum_{k=1}^{N} \left(\int_{\partial D_k} \frac{\partial u}{\partial n_k}(v-u_h)ds + \int_{D_k} (-\Delta u + cu)(v-u_h)dx \right) \\
&\quad - \int_D f(v-u_h)dx \\
&= \sum_{k=1}^{N} \sum_{l\in M(k)} \int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}[v-u_h]ds + \sum_{k=1}^{N} \int_{\partial D_k \cap \partial D} \frac{\partial u}{\partial n_k}(v-u_h)ds \\
&\quad + \int_D (-\Delta u + cu - f)(v-u_h)dx,
\end{aligned}
$$

where $\frac{\partial u}{\partial n_k}$ stands for the outward normal derivative of u on D_k and it is obvious that $\frac{\partial u}{\partial n_k} + \frac{\partial u}{\partial n_l} = 0$ on Γ_{lk}.

By noticing $[v] = v|_{D_l} - v|_{D_k} = 0$ on Γ_{lk}, we have

$$
\begin{aligned}
|a(u, v-u_h) - f(v-u_h)| \leq \sum_{k=1}^{N} \sum_{l\in M(k)} \left| \int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}[u_h]ds \right| \\
+ \sum_{k=1}^{N} \left| \int_{\partial D_k \cap \partial D} \frac{\partial u}{\partial n_k}(v-u_h)ds \right| + C\|v-u_h\|_{0,h}. \quad (3.7)
\end{aligned}
$$

By taking care of the second pair of square brackets of (3.6) similarly, we get

$$
\begin{aligned}
|a(u, v_h-u) - f(v_h-u)| \leq \sum_{k=1}^{N} \sum_{l\in M(k)} \left| \int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}[v_h]ds \right| \\
+ \sum_{k=1}^{N} \left| \int_{\partial D_k \cap \partial D} \frac{\partial u}{\partial n_k}(u-v_h)ds \right| + C\|u-v_h\|_{0,h}. \quad (3.8)
\end{aligned}
$$

Applying Cauchy–Schwarz inequality to the last term in (3.6) yields

$$
\begin{aligned}
\frac{1}{\alpha}|a(u-v_h, u_h-v_h)| &\leq \frac{M}{\alpha}\|u-v_h\|_{1,h}\|u_h-v_h\|_{1,h} \\
&\leq \frac{1}{2}\|u_h-v_h\|_{1,h}^2 + \frac{M^2}{2\alpha^2}\|u-v_h\|_{1,h}^2. \quad (3.9)
\end{aligned}
$$

Putting (3.7), (3.8), and (3.9) into (3.6), where $\frac{1}{2}\|u_h - v_h\|_{1,h}^2$ from (3.9) will be annihilated by the left hand side of (3.6), generates an upper bound of $\|u_h - v_h\|_{1,h}^2$. Then by putting it into (3.2) and noticing arbitrariness of $v \in V$ and $v_h \in V_h$, we prove (3.1) immediately by taking the infimum over $v \in V$ and $v_h \in V_h$. ■

The right hand side of (3.1) contains three terms

$$I_1 = C \inf_{v_h \in V_h} \left\{ \|u - v_h\|_{1,h}^2 + \|u - v_h\|_{0,h} + \sum_{k=1}^{N} \sum_{l \in M(k)} \left| \int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l} [v_h] ds \right| + \sum_{k=1}^{N} \left| \int_{\partial D_k \cap \partial D} \frac{\partial u}{\partial n_k} (u - v_h) ds \right| \right\},$$

$$I_2 = C \inf_{v \in V} \left\{ \|v - u_h\|_{0,h} + \sum_{k=1}^{N} \left| \int_{\partial D_k \cap \partial D} \frac{\partial u}{\partial n_k} (v - u_h) ds \right| \right\},$$

and

$$I_3 = C \sum_{k=1}^{N} \sum_{l \in M(k)} \left| \int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l} [u_h] ds \right|,$$

where I_1 is the approximation error, while I_2 and I_3 are the consistency errors. In the remainder of this section, we will show that $I_i \leq Ch^2$, $i = 1, 2, 3$.

Remark 3.1. The similarity and difference between the analysis of the linear equation and our variational inequality lie in the following facts: **1.** The first term in I_1 corresponds to the approximation error of the linear case while I_3 and the third term in I_1 are similar to the consistency error for the linear case; **2.** The terms with L^2-norm in I_1 and I_2 are generated by the nonlinear constraint and don't apply to the linear case. Meanwhile, the boundary terms in I_1 and I_2 have no counterparts for the linear case where homogeneous boundary conditions are assumed for simplicity. **3.** The construction of $v_h \in V$ and $v \in V$ should keep the nonnegative constraint so that we can not apply the discrete harmonic extension operator as in the linear case. Instead, a jump term needs to be constructed in the neighborhood of each nonmortar side.

A. Approximation Error

Firstly, let us estimate I_1. Construct $v_h \in X_h$ as follows: for all nodes P of Σ_{h_k}, $1 \leq k \leq N$,

$$v_h(P) = \begin{cases} u(P) & P \in N_k^0 \\ u(P) & P \in \bigcup_{l \in M(k)} N_{kl} \\ g(P) & P \in \partial D \\ (\Pi_{kl} I_l^h u)(P) & P \in \bigcup_{l \in S(k)} N_{kl} \end{cases} \tag{3.10}$$

It can be seen that $v_h = I_k^h u$ holds at all the nodes of Σ_{h_k} except on the nonmortar sides where the function values are defined as the mortar projection of those from the corresponding mortar sides. Figure 2 shows the mortar decomposition of D where the dashed lines denote the mortar sides. Here, Γ_{lk} stands for the mortar side, while Γ_{kl} is the nonmortar side. The dotted region S_{kl} denotes the union of all the triangles whose intersection with Γ_{kl} is either an edge or an interior node. The boundary of S_{kl} is then composed of Γ_{kl} and δ_{kl}, which is the portion inside D_k. Then $v_h = I_k^h u$ in any $\tau \in \Sigma_{h_k}$ outside S_{kl}.

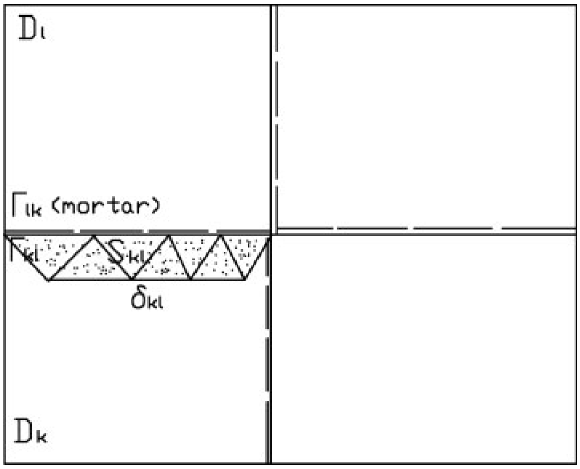

FIG. 2. Mortar decomposition of the domain.

It can be verified from (3.10) that $v_h \in V_h$. Therefore,

$$\begin{aligned}\|u - v_h\|_{0,h}^2 &= \sum_{k=1}^{N} \|u - v_h\|_{0,D_k}^2 \\ &\le C\left(\sum_{k=1}^{N} \left\|u - I_k^h u\right\|_{0,D_k}^2 + \sum_{k=1}^{N} \left\|I_k^h u - v_h\right\|_{0,D_k}^2\right) \\ &\le C\left(\sum_{k=1}^{N} h_k^4 \|u\|_{2,D_k}^2 + \sum_{k=1}^{N} \sum_{l\in S(k)} \left\|I_k^h u - v_h\right\|_{0,S_{kl}}^2\right). \qquad (3.11)\end{aligned}$$

The last term in (3.11) can be taken care as follows.

$$\begin{aligned}\left\|I_k^h u - v_h\right\|_{0,S_{kl}}^2 &= \sum_{\tau\in S_{kl}} \int_\tau \left(I_k^h u - v_h\right)^2 dx \\ &\le \sum_{\tau\in S_{kl}} \left\|I_k^h u - v_h\right\|_{0,\infty,\tau}^2 \cdot \int_\tau 1 dx \\ &\le \sum_{\tau\in S_{kl}} \left\|I_k^h u - v_h\right\|_{0,\infty,\tau\cap\Gamma_{kl}}^2 \cdot C_1 h_k^2 \\ &\le \sum_{\tau\in S_{kl}} \left(h_k^{-1}\right) \left\|I_k^h u - \Pi_{kl} I_l^h u\right\|_{0,2,\tau\cap\Gamma_{kl}}^2 \cdot C_1 h_k^2 \\ &= C_1 h_k \cdot \left\|\Pi_{kl}\left(I_k^h u - I_l^h u\right)\right\|_{0,\Gamma_{kl}}^2, \qquad (3.12)\end{aligned}$$

where the second inequality holds since $I_k^h u - v_h$ is a linear polynomial on any element $\tau \in S_{kl}$ and $I_k^h u - v_h = 0$ on δ_{kl} then $\max_\tau |I_k^h u - v_h|$ can only happen at nodes on Γ_{kl}, while the third inequality holds based on the inverse inequality.

By applying (2.9), the trace theorem [30], (2.7), and the mesh size uniformity assumption (2.4) to the last term on the right hand side of (3.12), successively, we have

$$\begin{aligned}\left\|\Pi_{kl}\left(I_k^h u - I_l^h u\right)\right\|_{0,\Gamma_{kl}}^2 &\le C_2 \left\|I_k^h u - I_l^h u\right\|_{0,\Gamma_{kl}}^2 \\ &\le C_2\left(\left\|u - I_k^h u\right\|_{0,\Gamma_{kl}}^2 + \left\|u - I_l^h u\right\|_{0,\Gamma_{kl}}^2\right) \\ &\le C_2\left(\left\|u - I_k^h u\right\|_{\frac{1}{2},D_k}^2 + \left\|u - I_l^h u\right\|_{\frac{1}{2},D_l}^2\right) \\ &\le C_2\left(h_k^3\|u\|_{2,D_k}^2 + h_l^3\|u\|_{2,D_l}^2\right) \\ &\le C_2 h_k^3. \end{aligned} \tag{3.13}$$

Combination of (3.12) and (3.13) yields

$$\left\|I_k^h u - v_h\right\|_{0,S_{kl}}^2 \le C_3 h_k^4. \tag{3.14}$$

Putting (3.14) into (3.11) yields an upper bound for the second term in I_1.

$$\|u - v_h\|_{0,h} \le Ch^2, \tag{3.15}$$

where C and C_i stand for generic constants independent of the mesh size h. Then the first term in I_1 proceeds as below.

$$\begin{aligned}\|u - v_h\|_{1,h}^2 &= \|u - v_h\|_{0,h}^2 + \sum_{k=1}^N |u - v_h|_{1,D_k}^2 \\ &\le \|u - v_h\|_{0,h}^2 + \sum_{k=1}^N \left|u - I_k^h u\right|_{1,D_k}^2 + \sum_{k=1}^N \left|I_k^h u - v_h\right|_{1,D_k}^2 \\ &\le C_1 h^4 + C_2 h^2 \|u\|_{2,D}^2 + C_3 \sum_{k=1}^N h_k^{-2} \left\|I_k^h u - v_h\right\|_{0,D_k}^2, \end{aligned} \tag{3.16}$$

where (3.15), (2.7) and the inverse inequality are applied. The last term in (3.16) can be estimated based on (3.14) as follows.

$$\begin{aligned}\sum_{k=1}^N h_k^{-2}\left\|I_k^h u - v_h\right\|_{0,D_k}^2 &= \sum_{k=1}^N h_k^{-2} \sum_{l\in S(k)} \left\|I_k^h u - v_h\right\|_{0,S_{kl}}^2 \\ &\le C_4 \sum_{k=1}^N h_k^{-2} \sum_{l\in S(k)} h_k^4 \\ &\le C_4 \sum_{k=1}^N h_k^2 \\ &\le C_4 h^2. \end{aligned} \tag{3.17}$$

Combination of (3.16) and (3.17) yields

$$\|u - v_h\|_{1,h}^2 \le Ch^2. \tag{3.18}$$

As to the third term in I_1, for $1 \le k \le N$, $l \in M(k)$, we apply (2.8) to obtain

$$\begin{aligned}
\left|\int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}[v_h]ds\right| &= \left|\int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}\left(\Pi_{lk} I_k^h u - I_k^h u\right)ds\right| \\
&= \left|\int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}\left(\left(I_l^h u - I_k^h u\right) - \Pi_{lk}\left(I_l^h u - I_k^h u\right)\right)ds\right| \\
&= \left|\int_{\Gamma_{lk}} \left(\frac{\partial u}{\partial n_l} - \psi\right)\left(\left(I_l^h u - I_k^h u\right) - \Pi_{lk}\left(I_l^h u - I_k^h u\right)\right)ds\right| \quad \forall \psi \in M_h(\Gamma_{lk}) \\
&\le \min_{\psi \in M_h(\Gamma_{lk})} \left\|\frac{\partial u}{\partial n_l} - \psi\right\|_{(H_{00}^{\frac{1}{2}}(\Gamma_{lk}))'} \cdot \left\|\left(I_l^h u - I_k^h u\right) - \Pi_{lk}\left(I_l^h u - I_k^h u\right)\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})},
\end{aligned} \tag{3.19}$$

where $(H_{00}^{\frac{1}{2}}(\Gamma_{lk}))'$ stands for the topological dual space of $H_{00}^{\frac{1}{2}}(\Gamma_{lk})$.

It is shown in [19] (Lemma 4.1, p. 37) that

$$\min_{\psi \in M_h(\Gamma_{lk})} \left\|\frac{\partial u}{\partial n_l} - \psi\right\|_{(H_{00}^{\frac{1}{2}}(\Gamma_{lk}))'} \le C_1 h \left\|\frac{\partial u}{\partial n_l}\right\|_{H^{\frac{1}{2}}(\Gamma_{lk})}. \tag{3.20}$$

Meanwhile, applying (2.9), the trace theorem and (2.7) to the last term in (3.19) yields

$$\begin{aligned}
\left\|\left(I_l^h u - I_k^h u\right) - \Pi_{lk}\left(I_l^h u - I_k^h u\right)\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})} &\le \left\|I_l^h u - I_k^h u\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})} \\
&\le \left\|u - I_l^h u\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})} + \left\|u - I_k^h u\right\|_{H_{00}^{\frac{1}{2}}(\Gamma_{lk})} \\
&\le \left\|u - I_l^h u\right\|_{H^1(D_l)} + \left\|u - I_k^h u\right\|_{H^1(D_k)} \\
&\le C_2 h \|u\|_{H^2(D)}.
\end{aligned} \tag{3.21}$$

By putting (3.20) and (3.21) into (3.19), we obtain

$$\left|\int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}[v_h]ds\right| \le Ch^2. \tag{3.22}$$

The last term in I_1 involves integrals on the boundary of D, $\gamma_k = \partial D_K \cap \partial D$ for $1 \le k \le N$. Define $p_k \in L^2(\gamma_k)$ as edge-wise constants on γ_k such that $p_k|_\sigma = \frac{1}{|\sigma|}\int_\sigma \frac{\partial u}{\partial n_k} ds$ for every edge $\sigma \subset \gamma_k$ and $|\sigma|$ denotes the length of σ. Triangle inequality implies that

$$\begin{aligned}
\left|\int_{\gamma_k} \frac{\partial u}{\partial n_k}(u - v_h)ds\right| &= \left|\int_{\gamma_k} \frac{\partial u}{\partial n_k}\left(u - I_k^h u\right)ds\right| \\
&\le \left|\int_{\gamma_k} \left(\frac{\partial u}{\partial n_k} - p_k\right)\left(u - I_k^h u\right)ds\right| + \left|\int_{\gamma_k} p_k\left(u - I_k^h u\right)ds\right|.
\end{aligned} \tag{3.23}$$

By applying Cauchy–Schwarz inequality, (2.7) and the trace theorem to the first term on the right hand side of (3.23), we have

$$\begin{aligned}\left|\int_{\gamma_k}\left(\frac{\partial u}{\partial n_k}-p_k\right)\left(u-I_k^h u\right)ds\right| &\le \left\|\frac{\partial u}{\partial n_k}-p_k\right\|_{0,\gamma_k}\cdot\left\|u-I_k^h u\right\|_{0,\gamma_k}\\ &\le C_1\cdot h_k^{\frac{1}{2}}\left\|\frac{\partial u}{\partial n_k}\right\|_{H^{\frac{1}{2}}(\gamma_k)}\cdot h_k^{\frac{3}{2}}\|u\|_{H^{\frac{3}{2}}(\gamma_k)}\\ &\le C_1 h_k^2\|\nabla u\|_{1,D_k}\|u\|_{2,D_k}\\ &\le C_1 h^2. \end{aligned}\tag{3.24}$$

Meanwhile, applying the trapezoidal integration rule to the last term in (3.23) yields

$$\begin{aligned}\left|\int_{\gamma_k} p_k\left(u-I_k^h u\right)ds\right| &= \left|\sum_{\sigma\subset\gamma_k}\int_\sigma p_k\left(u-I_k^h u\right)ds\right|\\ &= \left|\sum_{\sigma\subset\gamma_k} p_k\right|_\sigma \int_\sigma (u-I_k^h u)ds|\\ &\le \sum_{\sigma\subset\gamma_k}\left|\frac{1}{|\sigma|}\int_\sigma\frac{\partial u}{\partial n_k}ds\right|\cdot\left|\int_\sigma uds-\int_\sigma I_k^h uds\right|\\ &\le \sum_{\sigma\subset\gamma_k}\frac{1}{|\sigma|}\int_\sigma\left|\frac{\partial u}{\partial n_k}\right|ds\cdot C_2|\sigma|^3\\ &\le C_2 h_k^2\int_{\gamma_k}\left|\frac{\partial u}{\partial n_k}\right|ds\\ &\le C_2 h^2. \end{aligned}\tag{3.25}$$

By replacing the two terms on the right hand side of (3.23) with (3.24) and (3.25), we have

$$\left|\int_{\gamma_k}\frac{\partial u}{\partial n_k}(u-v_h)ds\right|\le Ch^2. \tag{3.26}$$

Combination of (3.15), (3.18), (3.22), and (3.26) implies that

$$I_1\le Ch^2. \tag{3.27}$$

B. Consistency Error

Next, let us consider the consistency errors I_2 and I_3. I_3 is estimated as follows. For $1 \le k \le N$, $l \in M(k)$,

$$
\begin{aligned}
\left|\int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}[u_h]ds\right| &= \left|\int_{\Gamma_{lk}} \frac{\partial u}{\partial n_l}(u_{h,D_l} - u_{h,D_k})ds\right| \\
&= \left|\int_{\Gamma_{lk}} \left(\frac{\partial u}{\partial n_l} - \psi\right)(u_{h,D_l} - u_{h,D_k})ds\right| \quad \forall \psi \in M_h(\Gamma_{lk}) \\
&\le \min_{\psi \in M_h(\Gamma_{lk})} \left\|\frac{\partial u}{\partial n_l} - \psi\right\|_{(H^{\frac{1}{2}}(\Gamma_{lk}))'} \|u_{h,D_l} - u_{h,D_k}\|_{H^{\frac{1}{2}}(\Gamma_{lk})} \\
&\le Ch \left\|\frac{\partial u}{\partial n_l}\right\|_{H^{\frac{1}{2}}(\Gamma_{lk})} \|u_{h,D_l} - u_{h,D_k}\|_{H^{\frac{1}{2}}(\Gamma_{lk})} \\
&\le Ch\|\nabla u\|_{1,D_l}\left(\|u - u_{h,D_l}\|_{H^{\frac{1}{2}}(\Gamma_{lk})} + \|u - u_{h,D_k}\|_{H^{\frac{1}{2}}(\Gamma_{lk})}\right) \\
&\le Ch(\|u - u_h\|_{1,D_l} + \|u - u_h\|_{1,D_k}),
\end{aligned}
$$

where (2.8) and the trace theorem are applied and the infimum is also bounded as in [18] (Sec. 4, p. 277). $(H^{\frac{1}{2}}(\Gamma_{lk}))'$ stands for the topological dual space of $H^{\frac{1}{2}}(\Gamma_{lk})$.

By putting it into the expression of I_3, we have

$$
\begin{aligned}
I_3 &\le Ch\|u - u_h\|_{1,h} \\
&\le Ch^2 + \frac{1}{4}\|u - u_h\|_{1,h}^2,
\end{aligned} \tag{3.28}
$$

where the last term $\frac{1}{4}\|u - u_h\|_{1,h}^2$ will be annihilated by $\|u - u_h\|_{1,h}^2$ on the left hand side of (3.1).

To evaluate I_2, we need to construct a particular function $v(x) \in V$, which is close enough to u_h under L^2 norm. It is obvious that $u_{h,D_k} \in P_1(D_k) \subset H^1(D_k)$ for every $1 \le k \le N$. However, u_h may not be continuous across each interface Γ_{kl} and $u_h = I_k^h g \ne g$ on $\gamma_k = \partial D_k \cap \partial D$. Therefore, v should be defined based on proper modification of u_h in the neighborhood of Γ_{kl} and γ_k, as shown in Fig. 3. S_{kl} and δ_{kl} have been specified in Fig. 2. The starred region S_{kk} denotes the union of all triangles whose intersection with γ_k is either an edge or a node. The boundary of S_{kk} is composed of γ_k and δ_{kk}, which is the portion inside D_k.

For every nonmortar side Γ_{kl} in D, $1 \le k \le N$, $l \in S(k)$, we define $v_{kl} \in H^2(S_{kl})$ as the harmonic function in S_{kl}:

$$
\begin{array}{ll}
\Delta v_{kl} = 0 & \text{in } S_{kl} \\
v_{kl} = u_{h,D_l} - u_{h,D_k} & \text{on } \Gamma_{kl} \\
v_{kl} = 0 & \text{on } \delta_{kl}
\end{array} \tag{3.29}
$$

Similarly, for every γ_k, $1 \le k \le N$, we define $v_{kk} \in H^2(S_{kk})$ as the harmonic function in S_{kk}:

$$
\begin{array}{ll}
\Delta v_{kk} = 0 & \text{in } S_{kk} \\
v_{kk} = g - I_k^h g & \text{on } \gamma_k \\
v_{kk} = 0 & \text{on } \delta_{kk}
\end{array} \tag{3.30}
$$

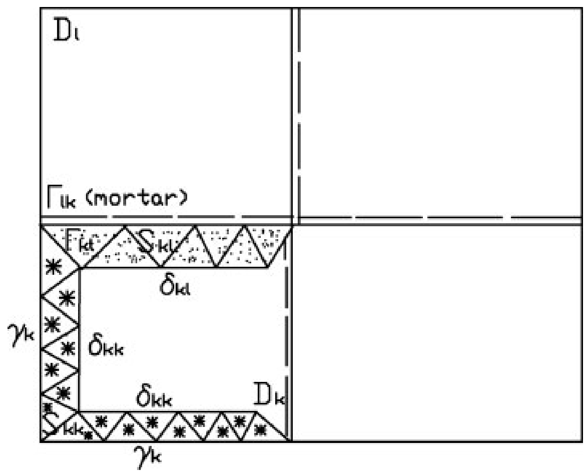

FIG. 3. Neighborhood of Γ_{kl} and γ_k.

$v(x)$ is now defined as follows: for $1 \leq k \leq N$,

$$v|_{D_k} = \begin{cases} u_{h,D_k} + v_{kl} & \text{in } S_{kl}, \;\; l \in S(k) \\ u_{h,D_k} + v_{kk} & \text{in } S_{kk} \\ u_{h,D_k} & \text{otherwise .} \end{cases} \tag{3.31}$$

Lemma 3.2. *v defined in* (3.31) *satisfies $v \in V$.*

Proof. It is obvious that both v_{kl} from (3.29) and v_{kk} from (3.30) belong to $H^1(D_k)$ under the assumption that they are set to zero outside S_{kl} and S_{kk}, respectively. Meanwhile, $u_{h,D_k} \in P_1(D_k) \subset H^1(D_k)$. Then (3.31) implies that $v|_{D_k} \in H^1(D_k)$. Furthermore, $v|_{D_k} = v|_{D_l}$ on Γ_{kl} for every pair of neighboring subdomains. Therefore, $v \in H^1(D)$.

Since u_{h,D_k} is a piecewise linear polynomial on S_{kl} and thus achieves its minimum on one of its nodes on ∂S_{kl}. The maximum principle [30] implies that v_{kl} achieves the minimum on ∂S_{kl} as well. Therefore, $v = u_{h,D_k} + v_{kl}$ achieves its minimum on ∂S_{kl} where either $v = u_{h,D_k} \geq 0$ on δ_{kl} or $v = u_{h,D_l} \geq 0$ on Γ_{kl}, i.e., $v \geq 0$ in S_{kl}.

Similarly, $v \geq 0$ in S_{kk}. In the other part of D_k, $v = u_{h,D_k} \geq 0$. Then $v \geq 0$ in D_k for $1 \leq k \leq N$.

Finally, (3.30) and (3.31) imply $v = g$ on γ_k for $1 \leq k \leq N$. Therefore, $v \in V$ holds. ■

The following lemmas reveal some important properties of $v_{kl}(x)$ in S_{kl} and $v_{kk}(x)$ in S_{kk}.

Lemma 3.3. *Suppose $D = [0, a] \times [0, h]$ is a 2D rectangle region and $u(x) \in H^1(D)$ satisfies $u(x_1, 0) = 0$, $0 \leq x_1 \leq a$, then*

$$\|u\|_{0,D} \leq Ch|u|_{1,D}.$$

Proof.

$$\begin{aligned}
\|u\|_{0,D}^2 &= \int_0^a \int_0^h u(x_1,x_2)^2 dx_2 dx_1 \\
&= \int_0^a \int_0^h (u(x_1,x_2) - u(x_1,0))^2 dx_2 dx_1 \\
&= \int_0^a \int_0^h \left(\int_0^{x_2} \frac{\partial u}{\partial x_2}(x_1,\eta) d\eta \right)^2 dx_2 dx_1 \\
&\leq \int_0^a \int_0^h \left(x_2 \int_0^h \left(\frac{\partial u}{\partial x_2}(x_1,\eta) \right)^2 d\eta \right) dx_2 dx_1 \\
&\leq \int_0^a \left(\int_0^h \left(\frac{\partial u}{\partial x_2}(x_1,\eta) \right)^2 d\eta \right) dx_1 \cdot \int_0^h x_2 dx_2 \\
&\leq \frac{1}{2} h^2 |u|_{1,D}^2
\end{aligned}$$

Therefore,

$$\|u\|_{0,D} \leq \frac{\sqrt{2}}{2} h |u|_{1,D}.]$$ ■

In fact, Lemma 3.3 can be extended to a general strip where the width is $O(h)$ and $u = 0$ holds along its bottom side, just as $v_{kl}(x)$ in S_{kl} and $v_{kk}(x)$ in S_{kk}.

Lemma 3.4.

$$\|v_{kl}\|_{H^1(S_{kl})} \leq C \|v_{kl}\|_{H^{\frac{1}{2}}(\Gamma_{kl})}, \quad \|v_{kk}\|_{H^1(S_{kk})} \leq C \|v_{kk}\|_{H^{\frac{1}{2}}(\gamma_k)}.$$

Proof. Application of Lemma 3.3, Green formula, and the trace theorem yields

$$\begin{aligned}
\|v_{kl}\|_{H^1(S_{kl})}^2 &\leq (1 + Ch^2) |v_{kl}|_{H^1(S_{kl})}^2 \\
&= C \left(\int_{\partial S_{kl}} \frac{\partial v_{kl}}{\partial n} v_{kl} ds - \int_{S_{kl}} \Delta v_{kl} \cdot v_{kl} dx \right) \\
&= C \int_{\Gamma_{kl}} \frac{\partial v_{kl}}{\partial n} v_{kl} ds \\
&\leq C \left\| \frac{\partial v_{kl}}{\partial n} \right\|_{(H^{\frac{1}{2}}(\Gamma_{kl}))'} \cdot \|v_{kl}\|_{H^{\frac{1}{2}}(\Gamma_{kl})} \\
&\leq C \left(\|v_{kl}\|_{H^1(S_{kl})}^2 + \|\Delta v_{kl}\|_{L^2(S_{kl})}^2 \right)^{\frac{1}{2}} \cdot \|v_{kl}\|_{H^{\frac{1}{2}}(\Gamma_{kl})} \\
&= C \|v_{kl}\|_{H^1(S_{kl})} \cdot \|v_{kl}\|_{H^{\frac{1}{2}}(\Gamma_{kl})},
\end{aligned}$$

i.e.,

$$\|v_{kl}\|_{H^1(S_{kl})} \leq C \|v_{kl}\|_{H^{\frac{1}{2}}(\Gamma_{kl})}.$$

The second inequality can be proved similarly. ■

Finally, let us consider I_2. For such $v \in V$ defined in (3.31), we have

$$\begin{aligned}\|v-u_h\|_{0,h}^2 &= \sum_{k=1}^{N}\|v-u_h\|_{0,D_k}^2\\ &= \sum_{k=1}^{N}\sum_{l\in S(k)}\|v-u_h\|_{0,S_{kl}}^2+\sum_{k=1}^{N}\|v-u_h\|_{0,S_{kk}}^2\\ &= \sum_{k=1}^{N}\sum_{l\in S(k)}\|v_{kl}\|_{0,S_{kl}}^2+\sum_{k=1}^{N}\|v_{kk}\|_{0,S_{kk}}^2. \end{aligned} \tag{3.32}$$

Lemma 3.3, Lemma 3.4, Cauchy-Schwarz inequality, and the trace theorem imply that

$$\begin{aligned}\|v_{kl}\|_{0,S_{kl}}^2 &\le Ch^2|v_{kl}|_{H^1(S_{kl})}^2\\ &\le Ch^2\|v_{kl}\|_{H^{\frac{1}{2}}(\Gamma_{kl})}^2\\ &= Ch^2\|u_{h,D_l}-u_{h,D_k}\|_{H^{\frac{1}{2}}(\Gamma_{kl})}^2\\ &\le Ch^2\left(\|u-u_{h,D_l}\|_{H^{\frac{1}{2}}(\Gamma_{kl})}^2+\|u-_{h,D_k}\|_{H^{\frac{1}{2}}(\Gamma_{kl})}^2\right)\\ &\le Ch^2\big(\|u-u_h\|_{1,D_l}^2+\|u-u_h\|_{1,D_k}^2\big). \end{aligned} \tag{3.33}$$

Similarly, we have

$$\begin{aligned}\|v_{kk}\|_{0,S_{kk}}^2 &\le Ch^2|v_{kk}|_{H^1(S_{kk})}^2\\ &\le Ch^2\|v_{kk}\|_{H^{\frac{1}{2}}(\gamma_k)}^2\\ &= Ch^2\left\|g-I_k^h g\right\|_{H^{\frac{1}{2}}(\gamma_k)}^2\\ &\le Ch^4\|g\|_{H^{\frac{3}{2}}(\gamma_k)}^2\\ &\le Ch^4\|u\|_{2,D_k}^2. \end{aligned} \tag{3.34}$$

Putting (3.33) and (3.34) into (3.32), we obtain

$$\|v-u_h\|_{0,h}^2 \le Ch^4+Ch^2\|u-u_h\|_{1,h}^2,$$

i.e.,

$$\|v-u_h\|_{0,h} \le Ch^2+Ch\|u-u_h\|_{1,h}. \tag{3.35}$$

From the definition of v in (3.31), we can see that the second term in I_2 is the same as the last term in I_1, which has been taken care in (3.26), i.e.,

$$\left|\sum_{k=1}^{N}\int_{\gamma_k}\frac{\partial u}{\partial n_k}(v-u_h)ds\right| \le Ch^2. \tag{3.36}$$

Therefore, (3.35) and (3.36) imply

$$\begin{aligned} I_2 &\leq Ch^2 + Ch\|u - u_h\|_{1,h} \\ &\leq Ch^2 + \frac{1}{4}\|u - u_h\|_{1,h}^2, \end{aligned} \tag{3.37}$$

where the last term $\frac{1}{4}\|u - u_h\|_{1,h}^2$ will be annihilated by $\|u - u_h\|_{1,h}^2$ on the left hand side of (3.1).

Combination of (3.27), (3.28), (3.37), and Lemma 3.1 yields the main result of this article as in Theorem 3.5.

Theorem 3.5. *Suppose $u \in V$ and $u_h \in V_h$ be solutions to* (2.3) *and* (2.6)*, respectively, and $u \in H^2(D)$. Then for $h = \max_{1\leq k\leq N} h_k$,*

$$\|u - u_h\|_{1,h} \leq Ch.$$

The following theorem shows pointwise convergence of the mortar solution u_h toward u on the skeleton S.

Theorem 3.6. *Suppose $u \in V$ and $u_h \in V_h$ be solutions to* (2.3) *and* (2.6)*, and $u \in H^2(D)$. Then for the common interface Γ_{kl} between any neighboring D_k and D_l,*

$$\|u_{h,D_k} - u\|_{0,\infty,\Gamma_{kl}} \leq Ch^{\frac{1}{2}}.$$

Proof. Triangle inequality implies that

$$\|u_{h,D_k} - u\|_{0,\infty,\Gamma_{kl}} \leq \left\|u - I_k^h u\right\|_{0,\infty,\Gamma_{kl}} + \left\|I_k^h u - u_{h,D_k}\right\|_{0,\infty,\Gamma_{kl}}. \tag{3.38}$$

By applying the inverse inequality, the trace theorem, the interpolation error estimate on Γ_{kl}, Theorem 3.5, and the fact $C \leq \frac{h_k}{h} \leq 1$ based on the uniformity assumption (2.4) to the second term on the right hand side, successively, we obtain

$$\begin{aligned} \left\|I_k^h u - u_{h,D_k}\right\|_{0,\infty,\Gamma_{kl}} &\leq C_1 h_k^{-\frac{1}{2}} \left\|I_k^h u - u_{h,D_k}\right\|_{0,\Gamma_{kl}} \\ &\leq C_1 h_k^{-\frac{1}{2}} \left(\left\|u - I_k^h u\right\|_{0,\Gamma_{kl}} + \|u - u_{h,D_k}\|_{0,\Gamma_{kl}}\right) \\ &\leq C_1 h_k^{-\frac{1}{2}} \left(h_k^{\frac{3}{2}} \|u\|_{H^{\frac{3}{2}}(\Gamma_{kl})} + \|u - u_h\|_{1,D_k}\right) \end{aligned}$$

$$\leq C_1 h_k \|u\|_{H^{\frac{3}{2}}(\Gamma_{kl})} + C_1 h_k^{-\frac{1}{2}} \cdot h$$
$$\leq C_1 h^{\frac{1}{2}}.$$

Similarly,

$$\left\|u - I_k^h u\right\|_{0,\infty,\Gamma_{kl}} \leq C_2 h_k^{\frac{3}{2}} h_k^{-\frac{1}{2}} \|u\|_{H^{\frac{3}{2}}(\Gamma_{kl})}$$
$$\leq C_2 h_k \|u\|_{2,D_k}$$
$$\leq C_2 h^{\frac{1}{2}}.$$

Putting the above two estimates into (3.38) yields the desired result. ∎

Remark 3.2. Based on Theorem 3.6, we can show u_h also satisfies the nonnegative constraint on the nonmortar sides when h is small enough. In fact, if the node P on nonmortar side Γ_{kl} is located within Ω where $u(P) > 0$, then $u_{h,D_k}(P) > 0$ holds when h is small enough; If P is located in $D \backslash \Omega$, since $\lim_{h\to 0} u_{h,D_k}(P) = 0$ and $u_{h,D_l} \geq 0$ on the mortar side Γ_{lk}, the orthogonality condition will imply $u_{h,D_k}(P) \geq 0$ on Γ_{kl} as well. As to the free seepage example in Section IV, we can always observe satisfaction of the nonnegative constraint on the whole domain for the provided mesh sizes.

IV. NUMERICAL EXAMPLES

A. Free Surface Seepage Example

As the first example of the mortar element method, we consider the problem of free surface seepage (Fig. 4): find the free surface in a steady, two-dimensional seepage through a rectangular dam.

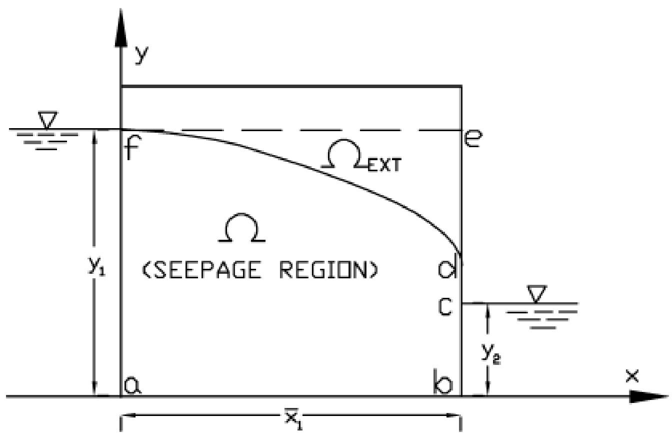

FIG. 4. Free boundary seepage problem.

In this study, the free surface, whose position is not known in advance, is to be found. In the seepage region Ω with $(x_1, x_2) = (x, y)$, the velocity potential ϕ must satisfy the following:

$$\begin{aligned} \phi &= 0 && \text{in } \Omega \\ \phi &= y_1 && \text{on } [af] \\ \phi &= y_2 && \text{on } [bc] \\ \phi &= y && \text{on } [cd] \\ \phi &= y && \text{on } \widehat{fd} \\ \phi_\eta &= 0 && \text{on } \widehat{fd} \\ \phi_\eta &= 0 && \text{on } [ab], \end{aligned} \tag{4.1}$$

where y_1 and y_2 are the heights of the water on the left and right sides, respectively. ϕ_η is the normal derivative of ϕ on $\partial\Omega$. The flow domain Ω is not known since the location of the free surface $\widehat{fd}$ needs to be found. Define D to be $D = \{(x, y) : 0 < x < \overline{x}_1, 0 < y < y_1\}$ and define $\overline{\phi}$ as an extension of ϕ as follows:

$$\overline{\phi} = \begin{cases} \phi(x, y) & \text{in } \Omega \\ y & \text{in } \overline{D} - \Omega = \Omega_{\text{ext}}. \end{cases} \tag{4.2}$$

Using the Baiocchi transformation [3], a new variable is defined as

$$w(x, y) = \int_y^{y_1} [\overline{\phi}(x, \overline{\eta}) - \overline{\eta}]d\overline{\eta}. \tag{4.3}$$

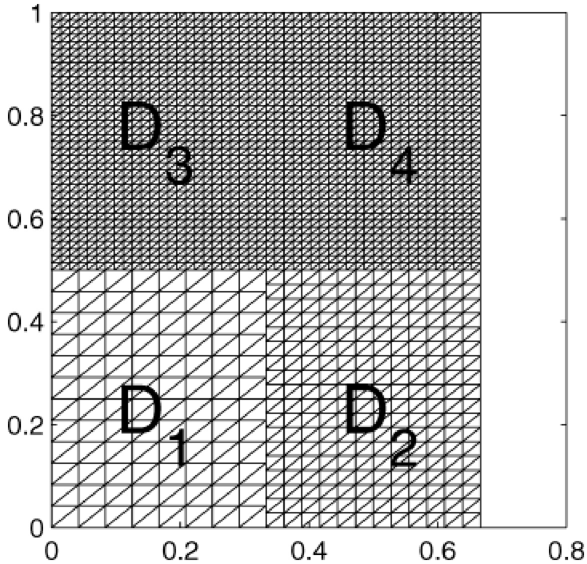

FIG. 5. Nonmatching decomposition of D.

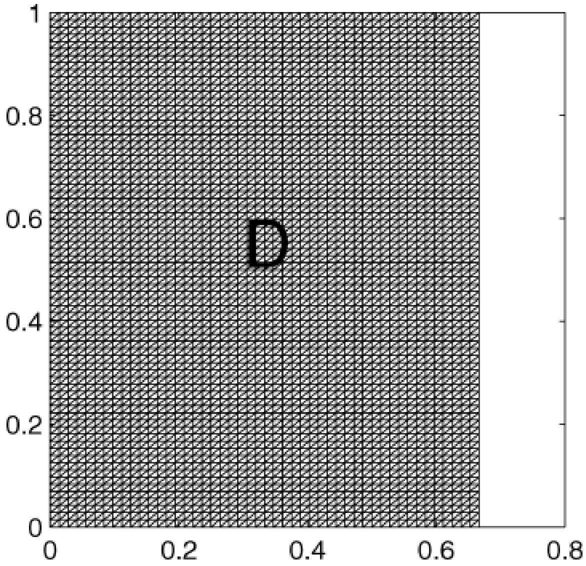

FIG. 6. Uniform mesh of D.

Then w satisfies

$$\Delta w = \chi_{\Omega} \quad \text{in } D$$
$$w = 0 \quad \text{in } \overline{D} - \overline{\Omega} \tag{4.4}$$

with the associated boundary conditions

$$w(0, y) = \frac{1}{2}(y_1 - y)^2 \quad \text{on } [af]$$

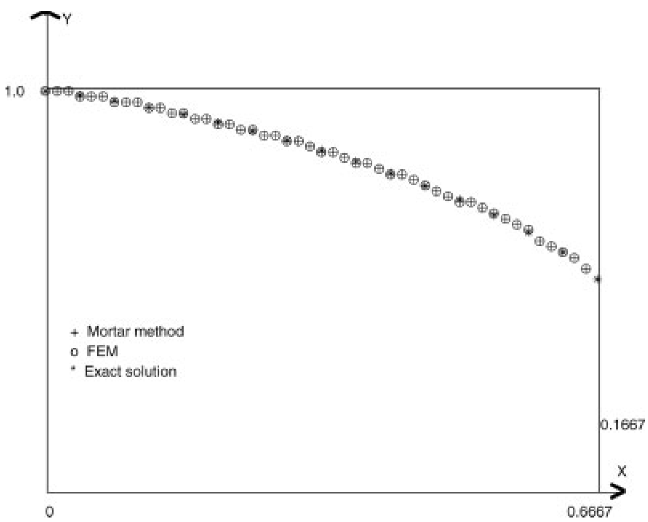

FIG. 7. Numerical results.

TABLE I. Free surface approximation.

x	0.083	0.167	0.250	0.333	0.416	0.500	0.583
$\lvert y_{\text{mortar}} - y_{\text{exact}} \rvert$	3.28 E −3	3.09 E −3	2.84 E −3	4.09 E −3	3.03 E −3	5.96 E −3	5.48 E −3
$\lvert y_{\text{conforming}} - y_{\text{exact}} \rvert$	3.28 E −3	3.09 E −3	2.84 E −3	4.09 E −3	3.03 E −3	5.96 E −3	5.48 E −3

$$w(x,0) = \frac{y_1^2}{2} - \frac{y_1^2 - y_2^2}{2\overline{x}_1}x \quad \text{on } [ab]$$

$$w(\overline{x}_1, y) = \frac{1}{2}(y_2 - y)^2 \quad \text{on } [bc]$$

$$w(x,y) = 0 \quad \text{on } [ce] \cup [fe]. \tag{4.5}$$

(4.4) and (4.5) are equivalent to the following variational inequality problem:

Find $w(x,y) \in H = \{u : u \in H^1(D), u|_{\partial D} \text{ satisfies (4.5)}\}$ such that

$$\int_D \nabla w \cdot \nabla(v - w)dxdy \geq \int_D -(v-w)dxdy \quad \forall v \in H. \tag{4.6}$$

The sample seepage problem has the following data: $y_1 = 1.00$, $y_2 = \frac{1}{6}$, $\overline{x}_1 = \frac{2}{3}$. To apply the mortar element method to solve (4.6), we decompose D into four equal nonoverlapping regions with different mesh sizes, as shown in Fig. 5. Since the free boundary is mainly located in D_3 and D_4 and far away from D_1 with possible intersection with D_2, a finer mesh is constructed in D_3 and D_4 to catch the free boundary more accurately and a coarser mesh is constructed in D_1 for computational efficiency, while the mesh size of D_2 is between those of the finer and coarser meshes. To this end, the mesh sizes of D_1, D_2, D_3, and D_4 are set to $\frac{1}{24}$, $\frac{1}{36}$, $\frac{1}{72}$, and $\frac{1}{72}$, respectively. The mortar is defined as the union of Γ_{21}, Γ_{24} from D_2 and Γ_{31}, Γ_{34} from D_3.

With the discretization space V_h constructed in (2.5), the mortart element solution $w_h \in V_h$ satisfies:

$$\sum_{k=1}^{4} \int_{D_k} \nabla w_h \cdot \nabla(v_h - w_h)dxdy \geq -\int_D (v_h - w_h)dxdy, \quad \forall v_h \in V_h. \tag{4.7}$$

We shall apply the relaxation method ([5], Chapter 5) to solve the discretized variational inequality in (4.7). The main idea of the relaxation method is to consider all active mesh nodes P_i, which include the internal nodes from all subdomains and the nodes on all mortar sides, sequentially at each step n. As we go through each active node P_i, we need to solve the following equation (4.8) for $w_h^n(P_i)$. When we solve $w_h^n(P_i)$, all the w_h^n values at other nodes are temporarily fixed. Then, we use the projected value $\max(w_h^n(P_i), 0)$ as the final choice of $w_h^n(P_i)$ at step n.

$$\int_D \nabla w_h^n \cdot \nabla \Phi_{P_i} dxdy = \int_D -\Phi_{P_i} dxdy, \tag{4.8}$$

where Φ_{P_i} is the nodal basis function of V_h such that $\Phi_{P_i}(P_j) = \delta_{ij}$.

TABLE II. Performance comparison.

	Number of unknowns	Number of iterations	Time spent (seconds)
Mortar element method	1964	386	252
Conforming FEM	3337	717	1582

TABLE III. Free surface approximation errors by mortar element method.

Mesh sizes	$\{\frac{1}{12}, \frac{1}{18}, \frac{1}{36}, \frac{1}{36}\}$	$\{\frac{1}{24}, \frac{1}{36}, \frac{1}{72}, \frac{1}{72}\}$	$\{\frac{1}{48}, \frac{1}{72}, \frac{1}{144}, \frac{1}{144}\}$
Free surface error	5.91 E −3	2.89 E −3	1.88 E −3

Convergence is determined when

$$\max_{P_i} \left| w_h^{n+1}(P_i) - w_h^n(P_i) \right| < \epsilon,$$

where $\epsilon = 1 \times 10^{-5}$ is some fixed positive constant.

The free surface is taken as the first mesh point with a value of w_h^n that is less than 1×10^{-6} when you move in the vertical direction for a fixed x. The final free surface location based on (4.7) is shown in Fig. 7 along with the exact solution reported by Crank [3], attributed to Polubarinova–Kochina.

To show the computational advantage of the mortar element method, we also solve (4.6) with conforming finite element method on the uniform mesh in D, as shown in Fig. 6. The approximated free surface location by conforming FEM is also reported in Fig. 7. The difference between the exact location of the free surface and the approximated ones by the mortar and conforming element methods are reported in Table I. We can see that the mortar element method has reached the same accuracy as conforming finite element method in approximating the exact solution of the seepage problem. These two programs are written with Matlab and run on Dell E520 Desktop with Intel Core(TM)2 Duo 1.86 GHz CPU and 1G RAM.

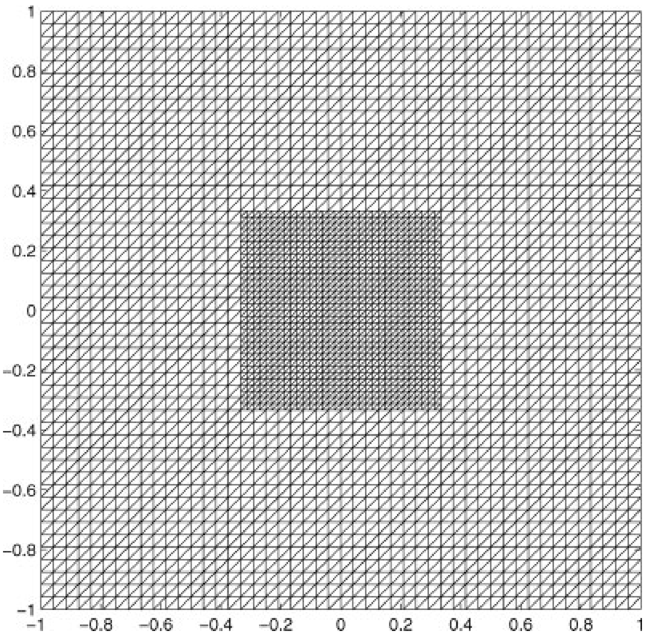

FIG. 8. Nonmatching meshes.

Table II compares the total number of unknowns (which is the same as number of active nodes), number of iterations required to reach the tolerance, and the time spent for both methods. We can see that the mortar element method can achieve the required precision with much less computational effort than conforming FEM due to its economical nonmatching technique.

Table III reports the discrete L^2-error between the exact location of the free boundary and the approximated location by mortar element method for different mesh sizes. The discrete error is defined as $\sqrt{\sum_{i=0}^{16} |y_{\text{mortar}}(x_i) - y_{\text{exact}}(x_i)|^2 \Delta x}$, where $\Delta x = \frac{1}{24}$ and $x_i = i \times \Delta x$. We can observe its linear dependence on decreasing mesh size h. Since w is constructed by the Baiocchi transformation and has no explicit formula within Ω, it is not convenient to evaluate $\|w - w_h\|_{H^1(\Omega)}$ with different mesh sizes. To this end, an obstacle problem is provided as the second numerical example where the exact solution has an explicit formula so that we can compare $\|u - u_h\|_{H^1(\Omega)}$ with respect to different values of h.

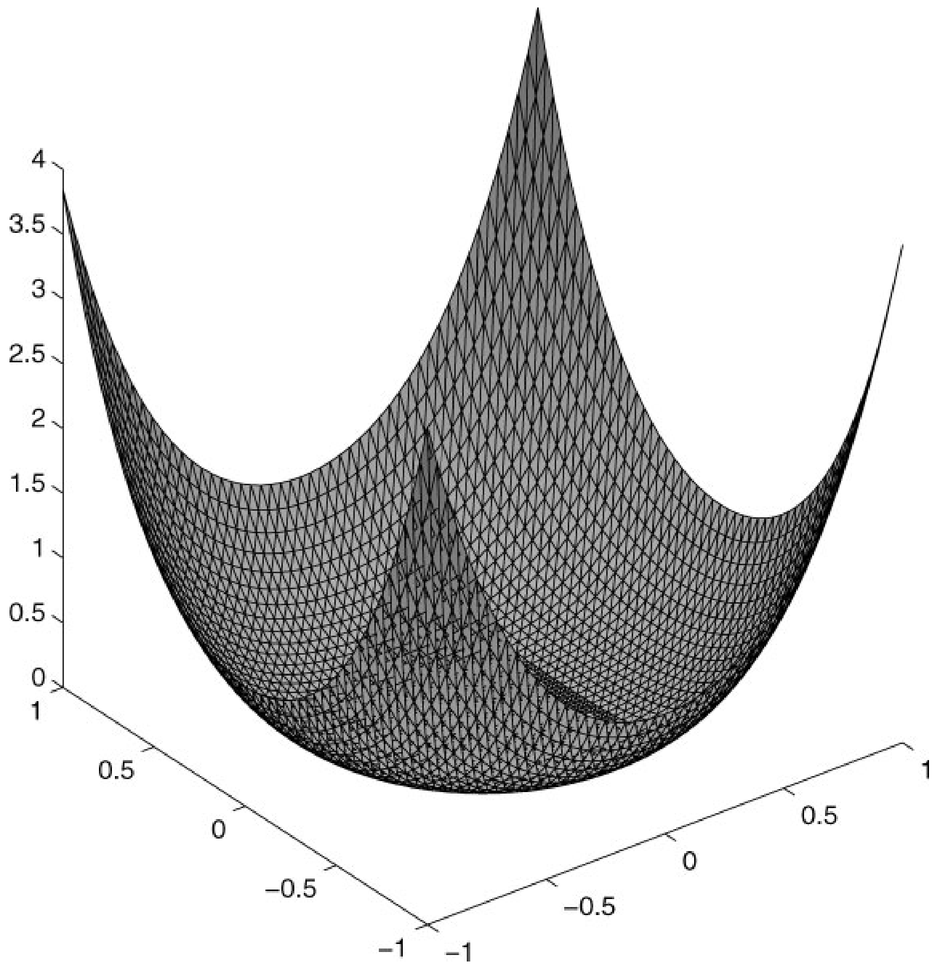

FIG. 9. Exact solution u.

B. Obstacle Problem

We consider the following obstacle problem with a constant obstacle $\chi = 0$ on the domain $\Omega = (-1,1) \times (-1,1)$:

Find $u \in K = \{v \in H^1(\Omega) | v \geq 0 \text{ in } \Omega,\ v = g \text{ on } \partial\Omega\}$ such that

$$a(u, v - u) \geq \int_\Omega f(v - u)\, dxdy, \tag{4.9}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dxdy$, Dirichlet boundary value is $g(x,y) = (x^2 + y^2 - r_0^2)^2$ with $r_0 = 0.2$, and

$$f(x,y) = \begin{cases} -8\big(2x^2 + 2y^2 - r_0^2\big) & x^2 + y^2 > r_0^2 \\ -8r_0^2\big(1 + r_0^2 - x^2 - y^2\big) & x^2 + r^2 \leq r_0^2. \end{cases}$$

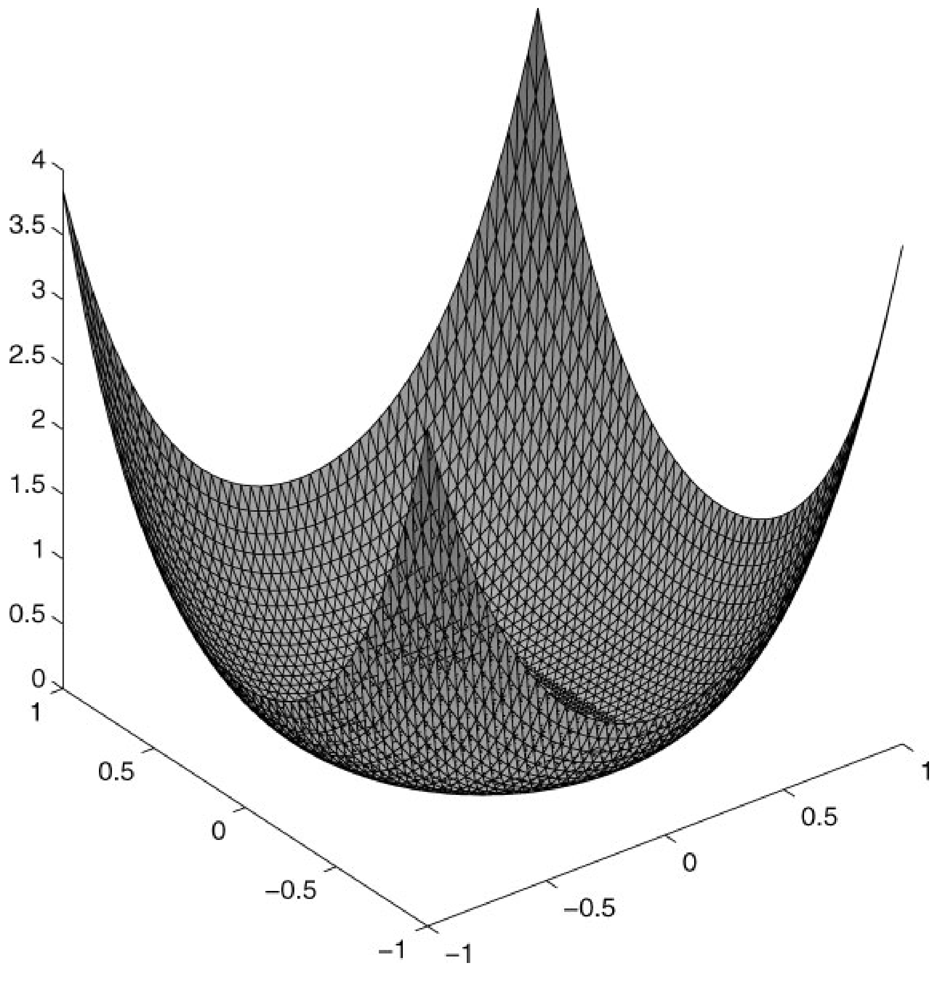

FIG. 10. Mortar solution u_h.

The exact solution to (4.9) reads

$$u(x, y) = \left(\max\left\{x^2 + y^2 - r_0^2, 0\right\}\right)^2.$$

To apply the mortar element method to solve (4.9), we decompose Ω into nine equal nonoverlapping regions shown in Fig. 8 where the mesh size of the central region $(-\frac{1}{3}, \frac{1}{3}) \times (-\frac{1}{3}, \frac{1}{3})$ is only half of that of the other regions, since the free boundary $x^2 + y^2 = r_0^2$ is only located within the central region.

We apply the same relaxation method as in the previous seepage problem with the stopping criterion $\epsilon = 1 \times 10^{-5}$. The mesh sizes of the central region and the other regions are set to $h_1 = \frac{1}{48}$ and $h_2 = \frac{1}{24}$, respectively. Figures 9 and 10 show the graphs of the exact solution u and the mortar element solution u_h, while Fig. 11 reports the error $|u(x, y) - u_h(x, y)|$ within Ω.

Finally, Table IV shows that $\|u - u_u\|_{H^1(\Omega)}$ linearly depends on decreasing mesh size h, i.e., $\|u - u_u\|_{H^1(\Omega)} = O(h)$, which conforms with the rate of convergence proved in Theorem 3.5. It is also interesting to notice that $\|u - u_h\|_{L^2(\Omega)} = O(h^2)$, which holds for conforming FEM as well. This indicates that the mortar element method can achieve the same error estimates as conforming FEM for general free boundary problems.

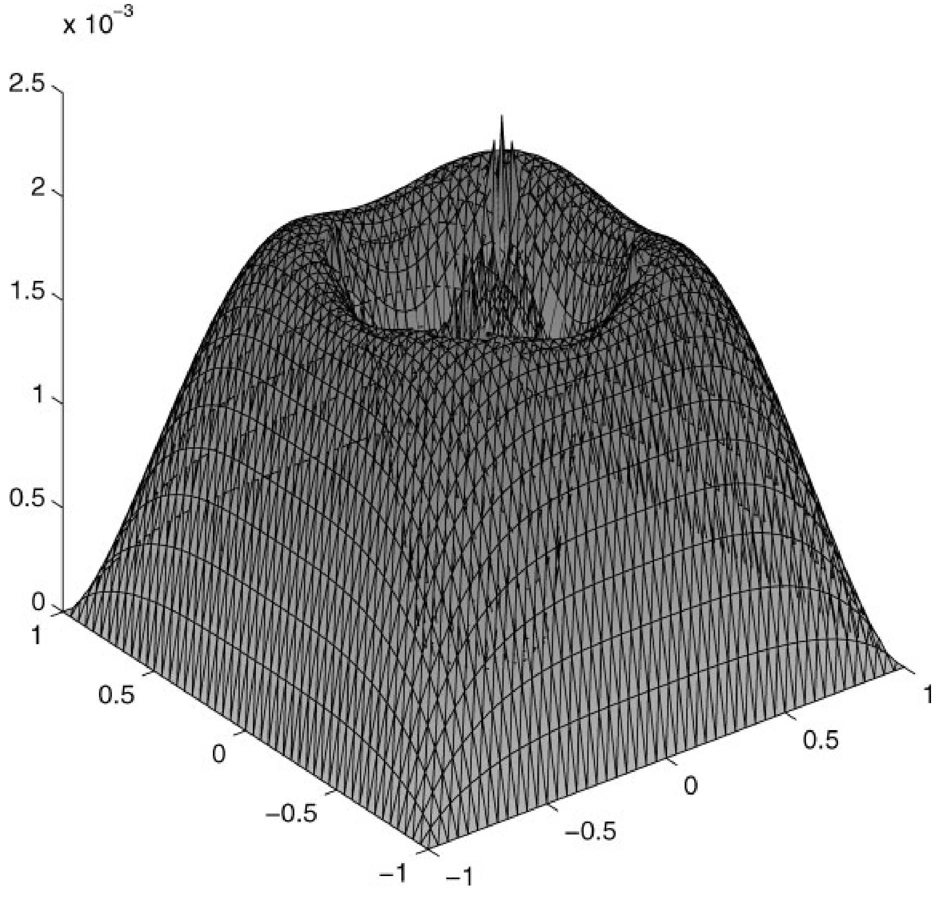

FIG. 11. $|u(x, y) - u_h(x, y)|$ in Ω.

TABLE IV. H1 and L2 errors for obstacle problem.

Mesh sizes $\{h_1, h_2\}$	$\{\frac{1}{12}, \frac{1}{6}\}$	$\{\frac{1}{24}, \frac{1}{12}\}$	$\{\frac{1}{48}, \frac{1}{24}\}$
$\|u - u_u\|_{H^1(\Omega)}$	66 E −2	3.2 E −2	1.6 E −2
$\|u - u_u\|_{L^2(\Omega)}$	3.0 E −2	9.4 E −3	2.5 E −3

V. CONCLUSION AND FUTURE WORK

In this article, we applied the mortar finite element method with standard Lagrange multiplier proposed in [18, 19] to solve the variational inequality of free boundary type. The original domain is split into multiple nonoverlapping subdomains, which are discretized arbitrarily with different mesh sizes. We showed that the mortar finite element method can yield the same error estimate as conforming P_1 element.

Mortar element method allows the coupling of different discretization schemes in different subregions and thus is superior in solving both linear and nonlinear heterogeneous problems. A large amount of work have been reported in the literature to solve linear problems with mortar element method while our work is concerned with nonlinear problems. Based on the work from this article, we will apply mortar element method to a wide range of nonlinear differential equations and variational inequalities, such as the problems with moving interface or boundaries. The error estimate of higher order mortar finite element methods for the variational inequalities will be considered. Meanwhile, we will investigate the possibility of applying mortar nonconforming finite element methods (such as Crouzeit–Raviart), which have less continuity requirement and thus provide more freedom of computation to those nonlinear problems.

References

1. A. Friedman, Variational principles and free boundary problems, Wiley, New York, 1982.
2. L. Badea and J. Wang, An additive Schwarz method for variational inequalities, Math Comput 69 (2000), 1341–1354.
3. J. Crank, Free and moving boundary problems, Clarendon Press, Oxford, 1984.
4. C. M. Elliott and J. R. Ockendon, Weak and variational methods for moving boundary problems, Research notes in mathematics 59, Pitman, Boston, 1982.
5. R. Glowinski, Numerical methods for nonlinear variational problems, Springer-Verlag, New York, 1984.
6. K. Hoffmann and J. Zou, Parallel algorithms of Schwarz variant for variational inequalities, Numer Funct Anal Optim 13 (1992), 449–462.
7. X. Tai, Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities, Numer Math 93 (2003), 755–786.
8. F. Brezzi, W. H. Hager, and P. A. Raviart, Error estimates for the finite element solution of variational inequalities, Numer Math 28 (1977), 431–443.
9. R. S. Falk, Error estimates for the approximation of a class of variational inequalities, Math Comput 28 (1974), 963–971.
10. C. M. Elliott, On the finite element approximation of an elliptic variational inequality arising from an implicit time discretization of the Stefan problem, IMA J Numer Anal 1 (1981), 115–125.
11. C. M. Elliott and J. Vladimr, An error estimate for a finite-element approximation of an elliptic variational inequality formulation of a Hele–Shaw moving-boundary problem, IMA J Numer Anal 1 (1983), 1–9.

12. Q. Deng, A nonoverlapping domain decomposition method for nonconforming finite element problems, Commun Pure Appl Anal 2 (2003), 295–306.

13. P. Lions, On the Schwarz alternating method III: a variant for nonoverlapping subdomains, T. F. Chan, R. Glowinski, J. Periaux, and O. B. Widlund, editors, Domain decomposition methods for partial differential equations, SIAM, Philadelphia, 1990, pp. 202–223.

14. L. Marini and A. Quarteroni, A relaxation procedure for domain decomposition methods using finite element, Numer Math 55 (1989), 575–598.

15. B. Jiang, J. C. Bruch Jr., and J. M. Sloss, A nonoverlapping domain decomposition method for variational inequalities derived from free boundary problems, Numer Methods Partial Differential Equations 22 (2006), 1–17.

16. B. Jiang, Convergence analysis of P1 finite element method for free boundary problems on nonoverlapping subdomains, Comput Methods Appl Mech Eng 196 (2006), 371–378.

17. B. Jiang and J. C. Bruch Jr., Numerical simulation of free seepage flow on non-matching grids, O. Widlund and D. Keyes, editors, Proceeding of the 16th International Conference on Domain Decomposition Methods, Springer Verlag, Berlin, 2007, pp. 603–610.

18. C. Bernardi, Y. Maday, and A. T. Patera, Domain decomposition by the mortar element method, H. G. Kaper, M. Garbey, and G. W. Pieper, editors, Asymptotic and numerical methods for partial differential equations with critical parameters, Kluwer, Dordrecht, The Netherlands, 1993, pp. 269–286.

19. C. Bernardi, Y. Maday, and A. T. Patera, A new nonconforming approach to domain decomposition: the mortar element method, H. Brezis and J.-L. Lions, editors, Nonlinear partial differential equations and their applications, Longman Scientific & Technical, Harlow, UK, 1994, pp. 13–51.

20. Y. Achdou, Y. Maday, and O. Widlund, Substructuring preconditioners for the Q_1 mortar element method, SIAM J Numer Anal 36 (1999), 551–580.

21. B. Belgacem, The mortar finite element method with Lagrange multipliers, Numer Math 84 (1999), 173–197.

22. M. Casarin and O. B. Widlund, A hierarchical preconditioner for the mortar finite element method, Electron Trans Numer Anal 4 (1996), 75–88.

23. P. Seshaiyer and M. Suri, Uniform hp convergence results for the mortar finite element method, Math Comput 69 (2000), 521–546.

24. P. Le Tallec and T. Sassi, Domain decomposition with nonmatching grids: augmented Lagrangian approach, Math Comput 64 (1995), 1367–1396.

25. B. Wohlmuth, An optimal a priori error estimate for nonlinear multibody contact problems, SIAM J Numer Anal 43 (2005), 156–173.

26. B. Belgacem, P. Hild, and P. Laborade, Extension of the mortar finite element method to a variational inequality modeling unilateral contact, Math Models Methods Appl Sci 9 (1999), 287–303.

27. P. Hild, The mortar finite element method for Bingham fluids, M2AN Math Model Numer Anal 35 (2001), 153–164.

28. J. L. Lions and G. Stampacchia, Variational inequalities, Commun Pure Appl Math 20 (1967), 493–519.

29. P. G. Ciarlet, The finite element method for elliptic problems, North-Holland Publishing Company, Amsterdam, 1979.

30. J. L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I, Springer-Verlag, New York, 1972.