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ENUMERATING CYCLIC QUASIPLATONIC GROUPS FOR A GIVEN SIGNATURE

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Abstract. It is an open problem to find how many topologically distinct ways that a Quasiplatonic group can act upon a surface X of genus $g(X) \geqslant 2$. We use the classification of cyclic Quasiplatonic groups to solve this counting problem in the case where the group in question is cyclic.

1. Introduction

A quasiplatonic surface is a compact Riemann surface, X, which admits a group of automorphisms, G, (called a quasiplatonic group) such that the quotient space, X/G, has genus 0 and the map $\pi_G : X \to X/G$ is branched over three points. Quasiplatonic groups acting on surfaces of genus 0 or 1 are well known. See, for example, [2]. A complete classification of Abelian quasiplatonic groups was given in [4] and [1]. It is not known, however, how many topologically distinct ways that an Abelian Quasiplatonic group can act upon a surface. There is a counting tool developed in [6] that we can use to answer this question for cyclic groups acting on quasiplatonic surfaces of genera greater than or equal to 2. We will develop formulae for each of the three cases of signatures for a cyclic group of order m, namely (m, m, m), (n, m, m) and (n_1, n_2, n_3).

2. Preliminaries

We begin with a section that contains the definition of a quasiplatonic surface as well as the properties of these surfaces that are relevant to our classification, much of which can be found in [5] or [6].

Definition 2.1. A finite group G is said to act **topologically** (in an orientation preserving manner) on a surface S of genus at least 2 if there is an injection $\varepsilon : G \to \text{Homeo}^+(S)$ into the group of orientation preserving homeomorphisms. Two actions ε_1, ε_2 are **topologically equivalent** if there is a homeomorphism h of S and an automorphism ω of G such that

$$\varepsilon_2(\omega(g)) = h \circ \varepsilon_1(g) \circ h^{-1}.$$

We will identify G with its image, and refer to each of its elements as an **automorphism** of the surface. (Note: We will also refer to group automorphisms as well. The context will make clear which we are referring to.)

For the remainder of this paper, unless otherwise states, all group actions on surfaces will be topological.

[1]This research was completed as part of the requirements for the Master's in Science in Mathematics at Portland State University.

Definition 2.2. Consider a space X and an automorphism group G acting on X. The **quotient space**, $X/G = \{g(x)|g \in G\}$, is the set of orbits of X under the action of G.

Definition 2.3. Let G be an automorphism group acting on a surface X. Let $\pi_G : X \to X/G$ be the function where every point in X is sent to its orbit. This map is known as the **natural quotient map**.

Definition 2.4. Let G be an automorphism group acting on a surface X and let $\pi_G : X \to X/G$ be the natural quotient map. The point $x \in X$ is a **ramification point** of π_G if there exists $g \in G$ such that $g \neq e$ and $g(x) = x$. ($Note$: We denote the identity element of G as e.)

Definition 2.5. If x is a ramification point, its image $\pi_G(x)$ is called a **branch point** of π_G.

Now that we have developed the language for surfaces that we will be using, we show an example that illustrates how these terms are used.

Example 2.6. Consider the surface of the sphere in $\mathbb{R}^3$, that is,

$$S^2 = \{(x, y, w) \in \mathbb{R}^3 | x^2 + y^2 + w^2 = 1\}.$$

Let $G = C_2 = \{e, g\}$, where for any $(x, y, w) \in S^2$, $e : (x, y, w) \to (x, y, w)$ and $g : (x, y, w) \to (-x, -y, w)$. By construction, G is an automorphism group acting on S^2, and all orbits not containing $n = (0, 0, 1)$ and $s = (0, 0, -1)$ are of the form $\{(x, y, w), (-x, -y, w)\}$. For n and s, $e(n) = g(n) = n$ and $e(s) = g(s) = s$. So the orbits of n and s respectively are $\{n\}$ and $\{s\}$. The set of all of these orbits is the quotient space S^2/G.

Clearly n and s are ramification points. It follows that the orbits $\{n\}$ and $\{s\}$ are branch points of π_G.

The most interesting aspect of the example above was the ramification points. In the example, there was two of them. The cases we are interested in is when there are exactly three ramification points. The most common examples of these types of group actions are the platonic solids under their full automorphism groups. In these cases, the ramification points are the vertices, the centers of the edges, and the centers of the faces. These partition nicely to form the three branch points in the quotient space. We now give formal definitions of quasiplatonic surfaces and quasiplatonic groups.

Definition 2.7. Let X be a compact Riemann surface. We call X a **quasiplatonic surface** if there exists a group G acting on X such that,

(i) X/G has genus 0, and
(ii) π_G is branched over 3 points exactly, where π_G is the natural quotient map from X to X/G.

We call the group G a **quasiplatonic group** and π_G a **quasiplatonic map**.

For the remainder of the paper, whether explicitly stated or not, all surfaces considered will be quasiplatonic surfaces.

The following definition is a simplification of a classification tool for certain types of group actions on surfaces. More can be found on Page 8 of [2]. We will use our simplification to distinguish topologically distinct group actions of a quasiplatonic group.

Definition 2.8. Suppose G is a quasiplatonic group acting on a quasiplatonic surface X such that X is branched over p_1, p_2, and p_3. Suppose $x_1, x_2, x_3 \in X$ are ramification points where $\pi_G(x_1) = p_1, \pi_G(x_2) = p_2$, and $\pi_G(x_3) = p_3$. Let $n_1 = |\operatorname{Stab}(x_1)|, n_2 = |\operatorname{Stab}(x_2)|$, and $n_3 = |\operatorname{Stab}(x_3)|$, and without loss of generality assume $n_1 \leqslant n_2 \leqslant n_3$. Then, the **signature** of (G, π_G) is the triple (n_1, n_2, n_3). We call the n_i the **periods** of the signature.

There are a limited and known number of quasiplatonic groups that can act on a surface of genus 1 or 0 quasiplatonically. A more detailed treatment of these can be found on page 9 of [2]. They are as follows:

Group	Signature	Genus	Group	Signature	Genus
D_n	$(2,2,n)$	0	C_6	$(2,3,6)$	1
A_4	$(2,3,3)$	0	C_4	$(2,4,4)$	1
S_4	$(2,3,4)$	0	C_3	$(3,3,3)$	1
A_5	$(2,3,5)$	0			

Since the genus 1 and 0 cases are completely classified, we will only concern ourselves with the $g(X) \geqslant 2$ case. The following theorem addresses this case specifically. While not explicitly used in proving the results of this paper, is a useful and common tool in this area of research. We have modified it slightly to fall in line with the rest of our notation. Also, at any point in the remainder of the paper when the genus of a surface is mentioned, note that it was found by using the formula below.

Theorem 2.9. *A group G is a quasiplatonic group for a surface X of genus $g(X) \geqslant 2$ with signature (n_1, n_2, n_3) if and only if $n_i \geqslant 2$, x and y generate G, $|x| = n_1$, $|y| = n_2$, and $|(xy)^{-1}| = n_3$, and*

$$g(X) = 1 - |G| + \frac{|G|}{2}(3 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}).$$

(The last condition is known as the Riemann-Hurwitz formula.)

Proof. For details, see Chapter 1 of [2]. □

This Theorem motivates the following definition.

Definition 2.10. Suppose (n_1, n_2, n_3) is a group signature. A triplet of group elements (x, y, z) in a finite group G is called a **Quasiplatonic generating vector of G for signature** (n_1, n_2, n_2) if $z = (xy)^{-1}$, and x, y and $(xy)^{-1}$ satisfy the conditions of **Theorem 2.9**.

We now look at an example of the previous theorem to show its application.

Example 2.11. Consider the group $G = C_{15} \times C_5$ where $< x >= C_{15}$ and $< y >= C_5$. Note that G is generated by elements x^3y and xy^4. This is apparent since their product is x^4, which generates C_{15}, and $(x^3y)(x^{12}) = y$, which generates C_5. The orders of x^3y, xy^4, and $((x^3)(xy^4))^{-1}$ are 5, 15, and 15, respectively. It follows from the above theorem that these elements generate G quasiplatonically with signature $(5, 15, 15)$ and genus 26. Further, $(x^3y, xy^4, ((x^3)(xy^4))^{-1})$ is a Quasiplatonic generating vector for this signature.

In theory, we could apply the same process to any group to find all signatures for a given group. In the abelian case, there is a simpler classification which can be found in [4] and [1]. Since this paper is concerned with the cyclic case only, we will only make use of the following theorem.

Theorem 2.12. *Fix a signature* (n_1, n_2, n_3) *and let* $M = \operatorname{lcm}(n_1, n_2, n_3)$*. There is a quasiplatonic surface* X *with quasiplatonic cyclic group* G *and signature* (n_1, n_2, n_3) *if and only if the following conditions are met:*

(*i*) $|G| = M = \operatorname{lcm}(n_1, n_2) = \operatorname{lcm}(n_1, n_3) = \operatorname{lcm}(n_2, n_3)$;
(*ii*) *if* M *is even, then exactly 2 of the periods* n_i *must be divisible by the maximum power of 2 that divides* $|G|$.

Proof. This is a specific case of **Harvey's Theorem**, proved in [4]. □

We will now look at the counting tool from [6] that we will use heavily in proving our result. We begin with a convenient definition.

Definition 2.13. Suppose (x, y, z) is a generating vector for a Quasiplatonic group G. Then we define the following permutations:

- $i_1 : x \to y, y \to x, z \to z$
- $i_2 : x \to x, y \to z, z \to y$
- $i_3 : x \to z, y \to y, z \to x$
- $j : x \to y, y \to z, z \to x$

For the remainder of the paper, when referring to i_1, i_2, i_3 or j, we are referring to the permutations above.

Theorem 2.14. *The number of topologically inequivalent Quasiplatonic generating vectors* T *with signature* (n_1, n_2, n_3) *on a quasiplatonic surface* S *can be calculated as follows.*

(*i*) *If all the* n_i *are distinct,*

$$T = \frac{|V_G|}{|\operatorname{Aut}(G)|}$$

where V_G *denotes the set of all quasiplatonic generating vectors of* G *with the given signature.*

(*ii*) *If* $n_2 = n_3$*, but* n_1 *is distinct, then*

$$T = \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{|\operatorname{Aut}(G)|}$$

where V_G *denotes the set of quasiplatonic generating vectors of* G *with the given signature for which the identification* i_2 *does not extend to an automorphism of* G*.* $V_{G,i}$ *denotes the set of quasiplatonic generating vectors of* G *with the given signature for which* i_2 *does extend to an automorphism of* G*. (We observe that it follows from Theorem 2.12 that it cannot be the case that* $n_1 = n_2$ *and are distinct from* n_3*.)*

(*iii*) *If* $n_1 = n_2 = n_3$ *then*

$$T = \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|}$$

where V_G *denotes the set of quasiplatonic generating vectors of* G *with the given signature for which the permutations* i_1*,* i_2*,* i_3 *and* j *do not extend to* *an automorphisms of G, $V_{G,i}$ denotes the set of quasiplatonic generating vectors of G with the given signature for which i_1, i_2 or i_3 does extend to an automorphism of G but j does not, $V_{G,j}$ denotes the set of quasiplatonic generating vectors of G with the given signature for which j does extend to an automorphism of G but i_1, i_2 and i_3 do not, and $V_{G,i,j}$ denotes the set of quasiplatonic generating vectors of G with the given signature for which i_1, i_2 or i_3 and j do extend to automorphisms of G.*

With this theorem, it is possible to find the number of inequivalent quasiplatonic vectors for any given group. But, this theorem relies heavily on the structure of the group in question. It is the aim of this paper to show that in the case of cyclic groups, we can simplify these formulas so that we only need to know the order of the group in question.

3. Enumerating Actions

We break our results into three cases based on the structure of the signature for a given group G. We begin with the case where each period of the signature is distinct. This case ends up having the nicest result due in large part to the fact that i_1, i_2, i_3 and j can never extend to automorphisms of G.

Theorem 3.1. *Consider a cyclic group G of order m, and fix a signature (n_1, n_2, n_3) where all the n_i are distinct. Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and the periods in in terms of these primes:*

$$m = \prod_{i=1}^{l} p_i^{k_i}, \qquad n_1 = \prod_{i=1}^{l} p_i^{r_i}, \qquad n_2 = \prod_{i=1}^{l} p_i^{s_i}, \qquad n_3 = \prod_{i=1}^{l} p_i^{t_i}.$$

We can reorder the p_i's and find an integer $w \leqslant l$ so that if $1 \leqslant i \leqslant w$, then r_i, s_i, and t_i are all equal to k_i, and if $w < i \leqslant l$, then exactly one of r_i, s_i, and t_i is less than k_i. In the latter case, let h_i represent this smaller value. Then, the number of inequivalent Quasiplatonic generating vectors T with signature (n_1, n_2, n_3) is

$$T = \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1} \varphi(p_i^{k_i}) \right) \left(\prod_{i=w+1}^{l} \varphi(p_i^{h_i}) \right),$$

where φ represents Euler's phi-function.

Proof. The existence of w, the reordering of the p_i's and the existence of the h_i's is a result of Theorem 2.12. By Theorem 2.14, we know that $T = \frac{|V_G|}{|\operatorname{Aut}(G)|}$. Since G is a cyclic group, we know that $|\operatorname{Aut}(G)| = \varphi(m)$. So, we only need to find $|V_G|$ to find T. This amounts to finding the number of valid Quasiplatonic generating vectors for this signature. To find this number, we will construct a valid Quasiplatonic generating vector for this signature. During each step in the process, we will count the number of choices we have.

Recall that G has $\varphi(m)$ generators. Choose one, and call it u. Observe that $G = C_{p_1^{k_1}} \times C_{p_2^{k_2}} \times \cdots \times C_{p_l^{k_l}}$ where $C_{p_i^{k_i}}$ is the cyclic group of order $p_i^{k_i}$. So, for each i, there exists an element $u_i \in G$ such that $u = \prod_{i=1}^{l} u_i$ and u_i generates $C_{p_i^{k_i}}$. We will use these generators to construct our vector, (x, y, z). Each of x, y and z will be a product of powers of the u_i, and we will count the number of choices we have for each i. Fix i.

Suppose exactly one of r_i, s_i, and t_i is less than k_i. Then, there are $\varphi(p_i^{h_i})$ choices of a_i such that $u_i^{a_i}$ is an element in $C_{p_i^{k_i}}$ of order $p_i^{h_i}$. For any such choice of a_i, we know that $u_i^{-(a_i+1)}$ has order $p_i^{k_i}$. Of the three elements u_i, $u_i^{a_i}$, and $u_i^{-(a_i+1)}$, let x_i be one whose order is the maximal power of p_i that divides n_1, and likewise for y_i with n_2 and z_i with n_3. The important thing to remember is that there were $\varphi(p_i^{h_i})$ choices for a_i, and therefore $\varphi(p_i^{h_i})$ choices for the elements x_i, y_i, and z_i.

The only other case to consider is when r_i, s_i, and t_i are all equal to k_i. Now we must choose a_i such that both $u_i^{a_i}$ and $u_i^{-(a_i+1)}$ have order $p_i^{k_i}$. So, p_i cannot divide a_i or $-(a_i+1)$. There are $\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})$ such choices. Now, label u_i, $u_i^{a_i}$ and $u_i^{-(a_i+1)}$ as x_i, y_i, and z_i, respectively. The important thing to remember is that there were $\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})$ choices for a_i, and therefore $\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})$ choices for the elements x_i, y_i, and z_i.

Now, let $x = \prod_{i=1}^{l} x_i$, $y = \prod_{i=1}^{l} y_i$, and $z = \prod_{i=1}^{l} z_i$. By construction, (x, y, z) is a valid generating vector. We also saw that the number of such vectors is

$$|V_G| = \varphi(m)\left(\prod_{i=1}^{w} \frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l} \varphi(p_i^{h_i})\right),$$

since there were $\varphi(m)$ choices for our generator u of G, and because we also found the number of choices for a_i in each case. Thus,

$$T = \left(\prod_{i=1}^{w} \frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l} \varphi(p_i^{h_i})\right)$$

since $|\operatorname{Aut}(G)| = \varphi(m)$.

□

We now illustrate this theorem with an example. We see that our newfound equation provides a quicker means for determining T.

Example 3.2. Consider the cyclic group G of order 105 with signature $(15, 21, 35)$ acting on a surface of genus 46. If we have a quasiplatonic vector $(x^a, x^b, x^{-(a+b)})$, then there are $\varphi(15) = 8$ choices for a and $\varphi(21) = 12$ choices for b. We know 7 will divide a. For a given choice of a, we need a b such that 5 divides b and 3 divides $a + b$. If $a \equiv 1 \mod 3$, then then there are 6 choices for b. Likewise, if $a \equiv 2 \mod 3$, then then there are also 6 choices for b. So, $|V_G| = 8 \times 6 = 48$. We also know that $|\operatorname{Aut}(G)| = \varphi(105) = 48$. So, Thereom 2.14 tells us that

$$T = \frac{|V_G|}{|\operatorname{Aut}(G)|} = \frac{48}{48} = 1.$$

Likewise, Theorem 3.1 tells us that

$$T = \left(\prod_{i=1}^{w} \frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l} \varphi(p_i^{h_i})\right) = \varphi(1)^3 = 1.$$

The following definitions will appear mysterious at the moment, but having these functions defined will prove extremely useful in the cases where i_2 or j extend to automorphisms.

Definition 3.3. We define $\tau_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $\tau_1(m,n)$ represents the number of nonzero noncongruent solutions x to $x^2 + 2x \equiv 0 \mod m$ where $\gcd(x,m) = \frac{m}{n}$. Likewise, define $\tau_2 : \mathbb{N} \to \mathbb{N}$ where $\tau_2(m)$ represents the number of nonzero noncongruent solutions x to $x^2 + x + 1 \equiv 0 \mod m$.

We now look at the case where exactly two of the periods must be identical. By **Harvey's Theorem**, we know that the two identical periods must be equal to the order of the group in question.

Theorem 3.4. *Consider a cyclic group G of order m, and fix a signature (n,m,m) where $n \neq m$. Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and n in in terms of these primes:*

$$m = \prod_{i=1}^{l} p_i^{k_i}, \qquad n = \prod_{i=1}^{l} p_i^{h_i}.$$

We can reorder the p_i's and find an integer $w \leqslant l$ so that if $1 \leqslant i \leqslant w$, then $h_i = k_i$, and if $w < i \leqslant l$, then $h_i < k_i$. Then, the number of inequivalent Quasiplatonic generating vectors T with signature (n,m,m) is

$$T = \frac{1}{2}\left(\tau_1(m,n) + \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l} \varphi(p_i^{h_i})\right)\right).$$

Proof. By Theorem 2.14, we know that $T = \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{|\operatorname{Aut}(G)|}$. Since G is a cyclic group, we know that $|\operatorname{Aut}(G)| = \varphi(m)$. So, we only need to find $|V_G|$ and $|V_{G,i}|$ to find T. We note that i_1 and i_3 cannot be extended to automorphisms. Let us examine the cases when i_2 is and is not an automorphism of G. Choose a generator $x \in G$ and suppose we choose a such that we have a quasiplatonic generating vector $(x^a, x^{-(a+1)}, x)$. Further, let us suppose that i_2 does extend to an automorphism. That is, the map that sends $x \to x^{-(a+1)}$, $x^{-(a+1)} \to x$, and $x^a \to x^a$ extends to an automorphism. Observe that

$$x^a = i_2(x^a) = (i_1(x))^a = (x^{-(a+1)})^a = x^{-a^2-a}$$

which tells us that

$$a^2 + 2a \equiv 0 \mod m.$$

Recall that $\tau_1(m,n)$ is the number of noncongruent solutions x to $x^2 + 2x \equiv 0 \mod m$ where $\gcd(x,m) = \frac{m}{n}$. i_2 extends to an automorphism if and only if a is such a solution. So, $|V_{G,i}| = \varphi(m)\tau_1(m,n)$. We also know by an argument similar to Theorem 3.1 that we can reorder the p_i's and find an integer $w \leqslant l$ so that if $1 \leqslant i \leqslant w$, then $k_i = h_i$, and if $w < i \leqslant l$, then $h_i < k_i$, and that

$$|V_G| + |V_{G,i}| = \varphi(m) \prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\varphi(p_i^{k_i}) \prod_{i=w+1}^{l} \varphi(p_i^{h_i}).$$

So,

$$|V_G| = \varphi(m) \prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\varphi(p_i^{k_i}) \prod_{i=w+1}^{l} \varphi(p_i^{h_i}) - \varphi(m)\tau_1(m,n).$$

Thus,

$$T = \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{|\operatorname{Aut}(G)|}$$

$$= \frac{\varphi(m)\prod_{i=1}^{w}\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\prod_{i=w+1}^{l}\varphi(p_i^{h_i}) - \varphi(m)\tau_1(m,n)}{2\varphi(m)} + \frac{\varphi(m)\tau_1(m,n)}{\varphi(m)}$$

$$= \frac{1}{2}\left(\tau_1(m,n) + \left(\prod_{i=1}^{w}\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l}\varphi(p_i^{h_i})\right)\right).$$

□

We again use an example to show the usefulness of our result.

Example 3.5. Consider the cyclic group G of order 120 with signature $(12, 120, 120)$ acting on a surface of genus 55. There are $\varphi(120)$ choices for a generator x of G. If we have a quasiplatonic vector $(x^a, x^{-(a+1)}, x)$, then there are $\varphi(12) = 4$ choices for a. These choices are 10, 50, 70 and 110. But, if $a = 50$ or 100, then $-(a+1) = 69$ or 9, respectively, which would contradict $|x^{-(a+1)}| = 120$. So, the only possible values of a are 10 or 70 which gives vector of (x^{10}, x^{109}, x) and (x^{70}, x^{49}, x). i_2 extends to an automorphism for both of these vectors, so Thereom 2.14 tells us that

$$T = \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{|\operatorname{Aut}(G)|}$$

$$= 0 + \frac{2\varphi(120)}{\varphi(120)} = 2.$$

The only solutions to $x^2 + 2x \equiv 0 \mod 15$ where $\gcd(x, 120) = \frac{120}{12} = 10$ are 10 and 70. So, $\tau_1(120, 12) = 2$. By Theorem 3.4, we see that

$$T = \frac{1}{2}\left(\tau_1(m,n) + \left(\prod_{i=1}^{w}\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l}\varphi(p_i^{h_i})\right)\right)$$

$$= \frac{1}{2}\left(2 + \left(\frac{3-2}{3-1}\varphi(3)\right)\varphi(2^2)\varphi(5^0)\right) = 2.$$

The following corollary is a special case of **Theorem 3.4** where our counting tool can be simplified by removing the τ_1 function. We need only make an observation about the prime factorization of n.

Corollary 3.6. *Consider a cyclic group G of order m, and fix a signature (n, m, m) where $n \neq m$. Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and n in in terms of these primes:*

$$m = \prod_{i=1}^{l} p_i^{k_i}, \qquad n = \prod_{i=1}^{l} p_i^{h_i}.$$

We can reorder the p_i's and find an integer $w \leqslant l$ so that if $1 \leqslant i \leqslant w$, then $k_i = h_i$, and if $w < i \leqslant l$, then $h_i < k_i$. If there is a prime $p_i \neq 2$ such that p_i divides n but $p_i^{k_i}$ does not divide n, then $\tau_1(m,n) = 0$. Further, the number of inequivalent Quasiplatonic generating vectors T with signature (n, m, m) is

$$T = \frac{1}{2}\left(\prod_{i=1}^{w}\frac{p_i-2}{p_i-1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l}\varphi(p_i^{h_i})\right).$$

Proof. Suppose we have a generating vector (x, y, z) of G with signature (n, m, m). Observe that x is a generator of G. So, there exists an integer a such that $z = x^a$ and $y = x^{-(a+1)}$. Suppose there is a prime $p_i \neq 2$ such that p_i divides n but $p_i^{k_i}$ does not divide n. It follows that p_i divides a but $p_i^{k_i}$ does not divide a. Recall from the proof of Theorem 3.4 that if i_2 extends to an automorphism that $a^2 + 2a \equiv 0 \mod m$, which means $a^2 + 2a \equiv 0 \mod p_i^{k_i}$. Since p_i divides a and $p_i \neq 2$, p_i cannot divide $a + 2$. So, since $p_i^{k_i}$ divides $a^2 + 2a = a(a + 2)$, then it follows that $p_i^{k_i}$ divides a, which is a contradiction. Thus, $\tau_1(m, n) = 0$. □

Example 3.7. Consider the cyclic group G of order p^k for some prime $p \neq 2$ with signature (p^h, p^k, p^k) where $1 \leqslant h < k$, acting on a surface of genus $\frac{1}{2}(p^k - p^{k-h})$. There are $\varphi(p^k)$ choices for a generator x of G. If we have a quasiplatonic vector $(x^a, x^{-(a+1)}, x)$, then are $\varphi(p^h)$ choices for a. So, the total number of quasiplatonic generating vectors of G is $|V_G| + |V_{G,i}| = \varphi(p^k)\varphi(p^h)$. But for any choice of a, $a^2 + 2a$ is not equivalent to $0 \mod p^k$, so $|V_{G,i}| = 0$. Thereom 2.14 tells us that

$$\begin{aligned} T &= \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{\operatorname{Aut}(G)|} \\ &= \frac{\varphi(p^k)\varphi(p^h)}{2\varphi(p^k)} + 0 = \frac{1}{2}\varphi(p^k). \end{aligned}$$

Likewise, Corollary 3.6 tells us that

$$\begin{aligned} T &= \frac{1}{2}\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\varphi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l} \varphi(p_i^{h_i})\right) \\ &= \frac{1}{2}\varphi(p^k). \end{aligned}$$

The last case to consider is the case where all of the periods are equal. It follows from **Harvey's Theorem** that the periods must all be the order of the group.

Theorem 3.8. *Consider a cyclic group G of order m, and fix a signature (m, m, m). Write m in its prime factorization:*

$$m = \prod_{i=1}^{l} p_i^{k_i}.$$

The number of inequivalent Quasiplatonic generating vectors T with signature (m, m, m) is

$$T = \frac{3 + 2\tau_2(m) + \varphi(m)\prod_{i=1}^{l} \frac{p_i - 2}{p_i - 1}}{6}$$

where φ represents Euler's phi-function.

Proof. By Theorem 2.14, we know that $T = \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|}$. Since G is a cyclic group, we know that $|\operatorname{Aut}(G)| = \varphi(m)$. We only need to find $|V_G|$, $|V_{G,i}|$, $|V_{G,j}|$ and $|V_{G,i,j}|$ to find T. We begin by finding when i_1, i_2, or i_3 is an automorphism. Since a vector where i_2 or i_3 extends to an automorphism is equivalent to a vector where i_1 extends to an automorphism, we will only concern ourselves with i_1. Choose a generator $x \in G$ and suppose we choose a such that we have a quasiplatonic generating vector $(x, x^{-(a+1)}, x^a)$. Further, let us suppose that i_1 does extend to an automorphism. That is, the map that sends $x \to x^{-(a+1)}$, $x^{-(a+1)} \to x$, and $x^a \to x^a$ extends to an automorphism. Observe that

$$x^a = i_1(x^a) = (i_1(x))^a = (x^{-(a+1)})^a = x^{-a^2-a}$$

which tells us that

$$a^2 + 2a \equiv 0 \mod m,$$

and further that for all p_i,

$$a^2 + 2a \equiv 0 \mod p_i^{k_i}.$$

We know that $\gcd(a, m) = 1$ since $|x^a| = m$. So, m cannot divide a, which means that m must divide $a+2$ since m divides a^2+2a. Thus, $a \equiv -2 \mod m$. Thus, the vector in question is (x, x, x^{-2}). Note that j cannot extend to an automorphism. So, $|V_{G,i}| = 3\varphi(m)$ and $|V_{G,i,j}| = 0$.

We now ask ourselves when j can extend to an automorphism. Choose a generator $x \in G$ and suppose we choose a such that we have a quasiplatonic generating vector $(x, x^{-(a+1)}, x^a)$. Further, let us suppose that j does extend to an automorphism. That is, the map that sends $x \to x^a$, $x^a \to x^{-(a+1)}$, and $x^{-(a+1)} \to x$ extends to an automorphism. Observe that

$$x^{-(a+1)} = j(x^a) = (j(x))^a = (x^a)^a = x^{a^2},$$

which tells us that

$$a^2 + a + 1 \equiv 0 \mod m.$$

Note that any solution to this congruence will be a value that is coprime to m, that is any such a will satisfy $|x^a| = m$. Recall that the number of solutions for a is $\tau_2(m)$. So, $|V_{G,j}| = \varphi(m)\tau_2(m)$.

By an argument similar to Theorem 3.1, we know that $|V_G| + |V_{G,i}| + |V_{G,j}| + |V_{G,i,j}| = \varphi(m)\prod_{i=1}^{l} \frac{p_i-2}{p_i-1}\varphi(p_i^{k_i}) = \varphi(m)^2 \prod_{i=1}^{l} \frac{p_i-2}{p_i-1}$. Solving for $|V_G|$, we get that

$$|V_G| = -3\varphi(m) - \varphi(m)\tau_2(m) + \varphi(m)^2 \prod_{i=1}^{l} \frac{p_i - 2}{p_i - 1}.$$

We now put all of the pieces together to see

$$T = \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|}$$

$$= \frac{-3\varphi(m) - \varphi(m)\tau_2(m) + \varphi(m)^2 \prod_{i=1}^{l} \frac{p_i-2}{p_i-1}}{6\varphi(m)} + \frac{3\varphi(m)}{3\varphi(m)} + \frac{\varphi(m)\tau_2(m)}{2\varphi(m)} + \frac{0}{\varphi(m)}$$

$$= \frac{-3 - \tau_2(m) + \varphi(m) \prod_{i=1}^{l} \frac{p_i-2}{p_i-1}}{6} + \frac{6}{6} + \frac{3\tau_2(m)}{6}$$

$$= \frac{3 + 2\tau_2(m) + \varphi(m) \prod_{i=1}^{l} \frac{p_i-2}{p_i-1}}{6}.$$

□

We now look at a couple of examples to illustrate the usefulness of our results.

Example 3.9. Consider the cyclic group G of order 21 with signature $(21, 21, 21)$ acting on a surface of genus 10. There are $\varphi(21)$ choices for a generator x of G. If we have a quasiplatonic vector $(x, x^a, x^{-(a+1)})$, then our choices for a are 1, 4, 10, 16, and 19. If we choose 1, 10 or 19, then i_1 or i_2 extends to an automorphism. If we choose 4 or 16, then j extends to an automorphism. Then, Thereom 2.14 tells us that

$$\begin{aligned} T &= \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|} \\ &= 0 + \frac{3\varphi(21)}{3\varphi(21)} + \frac{2\varphi(21)}{2\varphi(21)} + 0 = 2. \end{aligned}$$

Likewise, Theorem 3.8 tells us that

$$\begin{aligned} T &= \frac{3 + 2\tau_2(m) + \varphi(m)\prod_{i=1}^{l}\frac{p_i-2}{p_i-1}}{6} \\ &= \frac{3 + 4 + 12 \times \frac{5}{12}}{6} = 2. \end{aligned}$$

This was a somewhat simplistic example where in each generating vector, either i_1, i_2 or j extended to an automorphism. This need not be the case. In fact, it is possible that neither i_1, i_2 nor j will extend to automorphisms for the vast majority of generating vectors. The following examples illustrates this.

Example 3.10. Consider the cyclic group G of order 91 with signature $(91, 91, 91)$ acting on a surface of genus 45. There are $\varphi(91)$ choices for a generator x of G. If we have a quasiplatonic vector $(x, x^a, x^{-(a+1)})$, then there are a total of 55 choices for a. If we choose 1, 45 or 89, then i_1 or i_2 extends to an automorphism. If we choose 9, 16, 74 or 81, then j extends to an automorphism. For the other possible values of a, neither j, i_1, i_2 nor i_3 extends to an automorphism of G. So, Thereom 2.14 tells us that

$$\begin{aligned} T &= \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|} \\ &= \frac{48\varphi(91)}{6\varphi(91)} + \frac{3\varphi(91)}{3\varphi(91)} + \frac{4\varphi(91)}{2\varphi(91)} + 0 = 8 + 1 + 2 = 11. \end{aligned}$$

Likewise, Theorem 3.8 tells us that

$$\begin{aligned} T &= \frac{3 + 2\tau_2(m) + \varphi(m)\prod_{i=1}^{l}\frac{p_i-2}{p_i-1}}{6} \\ &= \frac{3 + 8 + 72 \times \frac{55}{72}}{6} = 11. \end{aligned}$$

4. In-depth Examples

Lastly, we will illustrate the full usefulness of our results by looking at two groups, and enumerating all the ways in which they can act quasiplatonically for each valid signature. The first example makes use of all four of the formulas that have been developed, while the second only uses the first three. This is because when the group has order m and m is even, the signature (m, m, m) is not possible. While the calculations are omitted, the author would like to assure the reader that using these formulas was far simpler than making use of Theorem 2.14.

Example 4.1. In the table below, we completely list the ways in which the cyclic group of order 315 can act quasiplatonically. That is, we list each valid signature, along with the appropriate genus and value of T.

Signature	Genus	T	Signature	Genus	T
$(315, 315, 315)$	157	8	$(45, 105, 315)$	153	6
$(105, 315, 315)$	156	15	$(9, 105, 315)$	139	2
$(63, 315, 315)$	155	8	$(45, 63, 315)$	152	1
$(45, 315, 315)$	154	5	$(15, 63, 315)$	145	2
$(35, 315, 315)$	153	8	$(5, 63, 315)$	124	1
$(21, 315, 315)$	150	5	$(35, 45, 315)$	150	3
$(15, 315, 315)$	147	3	$(21, 45, 315)$	147	2
$(9, 315, 315)$	140	2	$(7, 45, 315)$	132	1
$(7, 315, 315)$	135	3	$(9, 35, 315)$	136	1
$(5, 315, 315)$	126	2	$(15, 21, 315)$	140	2
$(3, 315, 315)$	105	1	$(45, 63, 105)$	151	2
$(63, 105, 315)$	154	10	$(35, 45, 63)$	148	1

Example 4.2. We now do the same for the cyclic group of order 360.

Signature	Genus	T	Signature	Genus	T
$(180, 360, 360)$	179	10	$(18, 120, 360)$	169	2
$(90, 360, 360)$	178	5	$(9, 120, 360)$	159	2
$(60, 360, 360)$	177	6	$(72, 90, 360)$	176	1
$(45, 360, 360)$	176	5	$(40, 90, 360)$	174	3
$(36, 360, 360)$	175	4	$(24, 90, 360)$	171	2
$(30, 360, 360)$	174	6	$(8, 90, 360)$	156	1
$(20, 360, 360)$	171	4	$(60, 72, 360)$	175	2
$(18, 360, 360)$	170	2	$(45, 72, 360)$	174	3
$(15, 360, 360)$	168	3	$(30, 72, 360)$	172	2
$(12, 360, 360)$	165	2	$(20, 72, 360)$	169	2
$(10, 360, 360)$	162	2	$(15, 72, 360)$	166	2
$(9, 360, 360)$	160	2	$(10, 72, 360)$	160	1
$(6, 360, 360)$	150	1	$(5, 72, 360)$	142	1
$(5, 360, 360)$	144	2	$(40, 45, 360)$	172	3
$(4, 360, 360)$	135	2	$(24, 45, 360)$	169	2
$(3, 360, 360)$	120	1	$(8, 45, 360)$	154	1
$(2, 360, 360)$	90	1	$(36, 40, 360)$	171	2
$(120, 180, 360)$	178	12	$(18, 40, 360)$	166	1
$(72, 180, 360)$	177	6	$(9, 40, 360)$	156	1
$(40, 180, 360)$	175	6	$(72, 120, 180)$	176	12
$(24, 180, 360)$	172	4	$(40, 72, 180)$	173	2
$(8, 180, 360)$	157	2	$(45, 72, 120)$	173	1
$(90, 120, 360)$	177	6	$(40, 72, 90)$	172	1
$(45, 120, 360)$	175	6	$(40, 45, 72)$	170	1
$(36, 120, 360)$	174	4			

The one glaring inefficiency to the formulas we have developed is the τ_1 and τ_2 functions. Ideally, we would like to have these functions written in terms of m or m's prime factorization, where m is the order of the group in question. This would be a potential area for further research. Another unanswered question is the number of ways in which a group (particularly a cyclic group) can act upon a surface of a particular genus, regardless of the signature. This is a natural extension of the work provided in this paper.

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