Enumerating Cyclic Quasiplatonic Groups For a Given Signature

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Preliminaries:

A finite group G is said to act **topologically** (in an orientation preserving manner) on a surface S if there is an injection $\epsilon : G \to \text{Homeo}^+(S)$ into the group of orientation preserving homeomorphisms. We will identify G with its image, and refer to each of its elements as an **automorphism** of the surface.

Example

A_4 acts topologically on the tetrahedron. That is, A_4 is a group of automorphisms that act on the tetrahedron. Recall that A_4 is the even permutations on a set of four letters. We can identify the vertices of the tetrahedron to be these letters.

Orbits

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The points with different orbit sizes:

Vertices - 4

Mid-points of edges - 6

Mid-points of faces - 4

Ramification Points

Definition: Given a surface X and an automorphism group G acting on X, if a point x on a surface X lies in an orbit that is not the largest orbit of points in X, then x is a ramification point.

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On the tetrahedron, the ramification points are:
- the vertices;
- the mid-points of the edges;
- the mid-points of the faces.

Quasiplatonic Surface

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Defintion: Given a surface X, if there exists an automorphism group G that acts on X such that G is a quasiplatonic group, then X is a quasiplatonic surface.

Signature of a quasiplatonic surface

Definition: Suppose G is a quasiplatonic group acting on a surface X such that X is a quasiplatonic surface. Suppose $x_1, x_2, x_3 \in X$ are ramification points lying in seperate orbits. Let

$$n_i = \frac{|G|}{|\operatorname{Orb}(x_i)|}, i = 1, 2, 3.$$

Then, the **signature** of (G, X) is the triple (n_1, n_2, n_3). We call the n_i the **periods** of the signature.

Signature of the Tetrahedron and A_4

Recall the size of the ramification orbits:
-Vertices - 4
-Mid-points of edges - 6
-Midpoints of faces - 4

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So, the signature of A_4 acting on this surface is:
$(\frac{|A_4|}{6}, \frac{|A_4|}{4}, \frac{|A_4|}{4}) = (\frac{12}{6}, \frac{12}{4}, \frac{12}{4}) = (2, 3, 3).$

General Theorem for Quasiplatonic Groups

Theorem: A group G is a quasiplatonic group for a surface X of genus $g(X)$ with signature (n_1, n_2, n_3) if and only if:

1) $n_i \geq 2$;

2) there exists $x, y \in G$ such that $|x| = n_1$, $|y| = n_2$, $|(xy)^{-1}| = n_3$ and $G =< x, y >$;

3) and $g(X) = 1 - |G| + \frac{|G|}{2}(3 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3})$.
(This formula is known as the Riemann-Hurwitz Formula.)

Application of General Theorem

Theorem: Recall that A_4 acting on the tetrahedron has signature $(2, 3, 3)$. We can now show this action is quasiplatonic:

1) Our periods 2 and 3 are both at least 2.

2) We choose elements $x = (12)(34)$ and $y = (123)$. So, $(xy)^{-1} = (234)$, and $|x| = 2$, $|y| = 3$, $|(xy)^{-1}| = 3$ and $A_4 =< x, y >$;

3) Lastly, we see $g(X) = 1 - 12 + \frac{12}{2}(3 - \frac{1}{2} - \frac{1}{3} - \frac{1}{3}) = 0.$

Generating Vectors

Defintion: Suppose (n_1, n_2, n_3) is a signature. A triplet of group elements (x, y, z) in a finite group G is called a **Quasiplatonic generating vector of G for signature** (n_1, n_2, n_2) if $z = (xy)^{-1}$, and x, y and $(xy)^{-1}$ satisfy the conditions of the previous theorem.

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Defintion: We consider two generating vectors (x_1, y_1, z_1) and (x_2, y_2, z_2) for a given group G and signature (n_1, n_2, n_2) to be **equivalent** if there exists $\sigma \in \mathrm{Aut}(G)$ such that $(\sigma(x_1), \sigma(y_1), \sigma(z_1)) = (x_2, y_2, z_2)$, or if (x_2, y_2, z_2) is a reordering of the elements of (x_1, y_1, z_1).

Generating Vectors

For our example of A_4 acting on the tetrahedron, we had generating vector $((12)(34), (123), (234))$.

An example of an equivalent generating vector is $((14)(23), (143), (243))$

Harvey's Theorem

Theorem: Fix a signature (n_1, n_2, n_3) and let $M = \mathrm{Lcm}(n_1, n_2, n_3)$. There is a quasiplatonic surface X with quasiplatonic cyclic group G and signature (n_1, n_2, n_3) if and only if the following conditions are met:

1) $|G| = M = \mathrm{Lcm}(n_1, n_2) = \mathrm{Lcm}(n_1, n_3) = \mathrm{Lcm}(n_2, n_3)$;

2) if M is even, then exactly 2 of the signature elements n_i must be divisible by the maximum power of 2 that divides $|G|$.

Results of Harvey's Theorem

Suppose we are considering the cyclic group of order m. Then, Harvey's Theorem tells us that there are only three types of signatures possible:

- (n_1, n_2, n_3) where each of the n_i are distinct,
- (n, m, m) where $n \neq m$, and
- (m, m, m).

Note that the final case can occur only when m is odd.

Wootton's Theorem: Part 1

Theorem: The number of inequivalent Quasiplatonic generating vectors T with signature (n_1, n_2, n_3) on a quasiplatonic surface X can be calculated as follows:

$$T = \frac{|V_G|}{|\operatorname{Aut}(G)|}$$

where V_G denotes the set of all quasiplatonic generating vectors of G with the given signature.

Results: Part 1

We will assume that $G = C_m$, and that we have a signature (n_1, n_2, n_3), where all the n_i are distinct. We know $T = \frac{|V_G|}{|\operatorname{Aut}(G)|}$ and that $|\operatorname{Aut}(G)| = \phi(m)$. So, we need only find $|V_G|$. That is, we need to count all of the valid quasiplatonic generating vectors for this case.

Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and the periods in terms of these primes: $m = \prod_{i=1}^{l} p_i^{k_i}, n_1 = \prod_{i=1}^{l} p_i^{r_i}, n_2 = \prod_{i=1}^{l} p_i^{s_i}, n_3 = \prod_{i=1}^{l} p_i^{t_i}$.

Results: Part 1

$G = C_{p_1^{k_1}} \times C_{p_2^{k_2}} \times \cdots \times C_{p_l^{k_l}}$. For each i, there exists an element $u_i \in G$ such that $u = \prod_{i=1}^{l} u_i$ and u_i generates $C_{p_i^{k_i}}$. We will use these generators to construct our vector, (x, y, z). Each of x, y and z will be a product of powers of the u_i, and we will count the number of choices we have for each i.

Results: Part 1

Let us first suppose that exactly one of r_i, s_i, and t_i is less than k_i. There are $\phi(p_i^{h_i})$ choices of a_i such that $u_i^{a_i}$ is an element in $C_{p_i^{k_i}}$ of order $p_i^{h_i}$. For any such choice of a_i, we know that $u_i^{-(a_i+1)}$ has order $p_i^{k_i}$. Of the three elements u_i, $u_i^{a_i}$, and $u_i^{-(a_i+1)}$, let x_i be one whose order is the maximal power of p_i that divides n_1, and likewise for y_i with n_2 and z_i with n_3. The important thing to remember is that there were $\phi(p_i^{h_i})$ choices for a_i, and therefore $\phi(p_i^{h_i})$ choices for the elements x_i, y_i, and z_i.

Results: Part 1

The other case to consider is $r_i = s_i = t_i = k_i$. Now we must choose a_i such that both $u_i^{a_i}$ and $u_i^{-(a_i+1)}$ have order $p_i^{k_i}$. So, p_i cannot divide a_i or $-(a_i + 1)$. There are $\frac{p_i-2}{p_i-1}\phi(p_i^{k_i})$ such choices. Now, label u_i, $u_i^{a_i}$ and $u_i^{-(a_i+1)}$ as x_i, y_i, and z_i, respectively. The important thing to remember is that there were $\frac{p_i-2}{p_i-1}\phi(p_i^{k_i})$ choices for a_i, and therefore $\frac{p_i-2}{p_i-1}\phi(p_i^{k_i})$ choices for the elements x_i, y_i, and z_i.

Results: Part 1

Now, let $x = \prod_{i=1}^{l} x_i$, $y = \prod_{i=1}^{l} y_i$, and $z = \prod_{i=1}^{l} z_i$. (x, y, z) is a valid generating vector. The number of choices for such vectors is:

$$|V_G| = \phi(m) \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1} \phi(p_i^{k_i}) \right) \left(\prod_{i=w+1}^{l} \phi(p_i^{h_i}) \right),$$

since there were $\phi(m)$ choices for our generator u of G, and because we also found the number of choices for a_i in each case.

Results: Part 1

Theorem: Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and the periods in terms of these primes: $m = \prod_{i=1}^{l} p_i^{k_i}, n_1 = \prod_{i=1}^{l} p_i^{r_i}, n_2 = \prod_{i=1}^{l} p_i^{s_i}, n_3 = \prod_{i=1}^{l} p_i^{t_i}$. We can reorder the p_i's and find an integer $w \leq l$ so that if $1 \leq i \leq w$, then r_i, s_i, and t_i are all equal to k_i, and if $w < i \leq l$, then exactly one of r_i, s_i, and t_i is less than k_i. In the latter case, let h_i represent this smaller value. Then,

$$T = \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1} \phi(p_i^{k_i}) \right) \left(\prod_{i=w+1}^{l} \phi(p_i^{h_i}) \right).$$

Counting Tools

Definition: Suppose (x, y, z) is a generating vector for a Quasiplatonic group G. Then we define the following permutations:

- $i_1 : x \longrightarrow y, y \longrightarrow x, z \longrightarrow z$
- $i_2 : x \longrightarrow x, y \longrightarrow z, z \longrightarrow y$
- $i_3 : x \longrightarrow z, y \longrightarrow y, z \longrightarrow x$
- $j : x \longrightarrow y, y \longrightarrow z, z \longrightarrow x$

Wootton's Theorem: Part 2

Theorem: The number of inequivalent Quasiplatonic generating vectors T with signature (n, m, m) on a quasiplatonic surface X can be calculated as follows:

$$T = \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{|\operatorname{Aut}(G)|}$$

where V_G denotes the set of quasiplatonic generating vectors of G with the given signature for which the identification i_2 does not extend to an automorphism of G. $V_{G,i}$ denotes the set of quasiplatonic generating vectors of G with the given signature for which i_2 does extend to an automorphism of G.

Results: Part 2

We will assume that $G = C_m$, and that we have a signature (n, m, m), where $n < m$. We know $T = \frac{|V_G|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{|\operatorname{Aut}(G)|}$ and that $|\operatorname{Aut}(G)| = \phi(m)$. So, we need only find $|V_G|$ and $|V_{G,i}|$. That is, we need to count all of the valid quasiplatonic generating vectors for this case.

Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and n in terms of these primes:

$m = \prod_{i=1}^{l} p_i^{k_i}, n = \prod_{i=1}^{l} p_i^{h_i}$.

Results: Part 2

We know by an argument similar to our first result that we can reorder the p_i's and find an integer $w \leq l$ so that if $1 \leq i \leq w$, then $k_i = h_i$, and if $w < i \leq l$, then $h_i < k_i$, and that $|V_G| + |V_{G,i}| = \phi(m) \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1} \phi(p_i^{k_i}) \right) \left(\prod_{i=w+1}^{l} \phi(p_i^{h_i}) \right)$. So, we need only find $|V_G|$ or $|V_{G,i}|$. We will find $|V_{G,i}|$. These are the vectors where i_2 does extend to an automorphism of G.

Results: Part 2

Choose a generator $x \in G$ choose a such that we have a quasiplatonic generating vector $(x^a, x^{-(a+1)}, x)$where i_2 extends to an automorphism of G. That is, the map that sends $x \to x^{-(a+1)}$, $x^{-(a+1)} \to x$, and $x^a \to x^a$ extends to an automorphism. Observe that

$$x^a = i_2(x^a) = (i_1(x))^a = (x^{-(a+1)})^a = x^{-a^2-a}$$

which tells us that

$$a^2 + 2a \equiv 0 \mod m.$$

Results: Part 2

Definition: We define $\tau_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $\tau_1(m,n)$ represents the number of nonzero noncongruent solutions a to $a^2 + 2a \equiv 0 \mod m$ where $\gcd(a,m) = \frac{m}{n}$.

i_2 extends to an automorphism if and only if a is such a solution. So, $|V_{G,i}| = \phi(m)\tau_1(m,n)$.

Results: Part 2

Theorem: Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m. Write m and n in terms of these primes: $m = \prod_{i=1}^{l} p_i^{k_i}, n = \prod_{i=1}^{l} p_i^{h_i}$. We can reorder the p_i's and find an integer $w \leq l$ so that if $1 \leq i \leq w$, then $h_i = k_i$, and if $w < i \leq l$, then $h_i < k_i$. Then, the number of inequivalent Quasiplatonic generating vectors T with signature (n, m, m) is $T = \frac{1}{2}\left(\tau_1(m,n) + \left(\prod_{i=1}^{w} \frac{p_i-2}{p_i-1}\phi(p_i^{k_i})\right)\left(\prod_{i=w+1}^{l} \phi(p_i^{h_i})\right)\right).$

Wootton's Theorem: Part 3

Theorem: The number of inequivalent Quasiplatonic generating vectors T with signature (n, m, m) on a quasiplatonic surface X can be calculated as follows:

$$T = \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|}$$

where...

Results: Part 3

We will assume that $G = C_m$, and that we have a signature (m, m, m). We know $T = \frac{|V_G|}{6|\operatorname{Aut}(G)|} + \frac{|V_{G,i}|}{3\operatorname{Aut}(G)|} + \frac{|V_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|V_{G,i,j}|}{|\operatorname{Aut}(G)|}$ and that $|\operatorname{Aut}(G)| = \phi(m)$. So, we need only find $|V_G|$,$|V_{G,i}|$, $|V_{G,j}|$ and $|V_{G,i,j}|$.

Results: Part 3

Let $p_1, p_2, ..., p_l$ be the distinct primes that divide m and write $m = \prod_{i=1}^{l} p_i^{k_i}$.

By an argument similar to the first case, we know that $|V_G| + |V_{G,i}| + |V_{G,j}| + |V_{G,i,j}| = \phi(m) \prod_{i=1}^{l} \frac{p_i - 2}{p_i - 1} \phi(p_i^{k_i}) = \phi(m)^2 \prod_{i=1}^{l} \frac{p_i - 2}{p_i - 1}$.

Results: Part 3

We begin by finding when i_1, i_2, or i_3 is an automorphism. Since a vector where i_2 or i_3 extends to an automorphism is equivalent to a vector where i_1 extends to an automorphism, we will only concern ourselves with i_1. Choose a generator $x \in G$ and suppose we choose a such that we have a quasiplatonic generating vector $(x, x^{-(a+1)}, x^a)$. Further, let us suppose that i_1 does extend to an automorphism. That is, the map that sends $x \to x^{-(a+1)}$, $x^{-(a+1)} \to x$, and $x^a \to x^a$ extends to an automorphism.

Results: Part 3

Observe that

$$x^a = i_1(x^a) = (i_1(x))^a = (x^{-(a+1)})^a = x^{-a^2-a}$$

which tells us that

$$a^2 + 2a \equiv 0 \mod m.$$

We know that $\gcd(a, m) = 1$ since $|x^a| = m$. So, m cannot divide a, which means that m must divide $a + 2$ since m divides $a^2 + 2a$. Thus, $a \equiv -2 \mod m$. Thus, the vector in question is (x, x, x^{-2}). Note that in this case j cannot extend to an automorphism. So, $|V_{G,i}| = 3\phi(m)$ and $|V_{G,i,j}| = 0$.

Results: Part 3

Now we suppose that j does extend to an automorphism. That is, the map that sends $x \to x^a$, $x^a \to x^{-(a+1)}$, and $x^{-(a+1)} \to x$ extends to an automorphism. Observe that

$$x^{-(a+1)} = j(x^a) = (j(x))^a = (x^a)^a = x^{a^2},$$

which tells us that

$$a^2 + a + 1 \equiv 0 \mod m.$$

Results: Part 3

Definition: We define $\tau_2 : \mathbb{N} \rightarrow \mathbb{N}$ where $\tau_2(m)$ represents the number of nonzero noncongruent solutions x to $x^2 + x + 1 \equiv 0 \mod m$.

Note that any solution to this congruence will be a value that is coprime to m, that is any such a will satisfy $|x^a| = m$. So, any solution to the congruence will create a valid generating vector. Thus, $|V_{G,j}| = \phi(m)\tau_2(m)$.

Results: Part 3

Theorem: Write m in its prime factorization: $m = \prod_{i=1}^{l} p_i^{k_i}$. The number of inequivalent Quasiplatonic generating vectors T with signature (m, m, m) is

$$T = \frac{3 + 2\tau_2(m) + \phi(m) \prod_{i=1}^{l} \frac{p_i - 2}{p_i - 1}}{6}.$$