Day 14

Definition: Let ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) be groups. $\boldsymbol{G}$ is *isomorphic* to $\boldsymbol{H}$ if there exists a bijective function $\varphi:\boldsymbol{G}\to\boldsymbol{H}$ such that

$$\forall a, b \in \boldsymbol{G},\ \varphi(a\bullet b) = \varphi(a)*\varphi(b).$$

In this case, the function ϕ is said to be an *isomorphism*.

Now let's prove some theorems concerning isomorphisms!

Theorem: Suppose ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) are groups and there exists an isomorphism $\varphi:\boldsymbol{G}\to\boldsymbol{H}$. Then there exists an isomorphism $\psi:\boldsymbol{H}\to\boldsymbol{G}$.

Theorem: Suppose ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) are isomorphic groups. If $\boldsymbol{G}$ is abelian then $\boldsymbol{H}$ is abelian.

Theorem: Suppose ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) are groups and φ: $\boldsymbol{G}\to\boldsymbol{H}$ is an isomorphism. Let e_G, e_H denote the identity elements of these groups. Then $\varphi(e_G) = e_H$.

Theorem: Suppose ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) are groups and φ: $\boldsymbol{G}\to\boldsymbol{H}$ is an isomorphism. Let $a \in \boldsymbol{G}$, then $\varphi(a)^{-1} = \varphi(a^{-1})$.

Theorem: Suppose ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) are groups and ϕ: $\boldsymbol{G}\to\boldsymbol{H}$ is an isomorphism. Let $a \in \boldsymbol{G}$ and $n\in \boldsymbol{N}$, then $\varphi(a)^n = \varphi(a^n)$.

This is a proof by mathematical induction. We want to show it is true for any integer. We do this in two steps.

Base Case: Prove it works for $n = 1$.

Induction Step: Prove that IF it works for $n = k$, THEN it works for $n = k + 1$.

Together these imply it works for any $n\in \boldsymbol{N}$.

Two "leftover" results for homework:

Theorem: Suppose $\boldsymbol{G}$ and $\boldsymbol{H}$ are isomorphic groups then, if $\boldsymbol{G}$ is cyclic, then $\boldsymbol{H}$ is cyclic. (Recall a group is cyclic if there exists an element **a** such that for every element **g** in $\boldsymbol{G}$, there exists some integer **i** such that $\mathbf{g}=\mathbf{a}^{\mathbf{i}}$.)

Theorem: Suppose ($\boldsymbol{G}$, $\bullet$) and ($\boldsymbol{H}$, $*$) are groups and φ: $\boldsymbol{G}\to\boldsymbol{H}$ is an isomorphism. Let $a \in \boldsymbol{G}$. Then, $|\boldsymbol{\varphi(a)}| = |\boldsymbol{a}|$. (Recall the order of an element is the least positive integer **i** such that $\mathbf{a}^{\mathbf{i}}$ is the identity).