

# Graphic Realizations of Sequences

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Under the direction of  
Dr. John S. Caughman

June 6, 2011

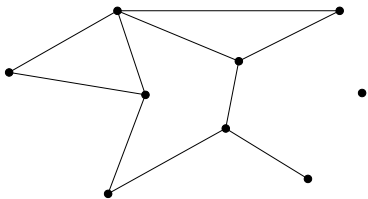
# Introduction

This talk is an overview of results developed in the paper “Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size” written by Jian-Hua Yin and Jiong-Sheng Li.

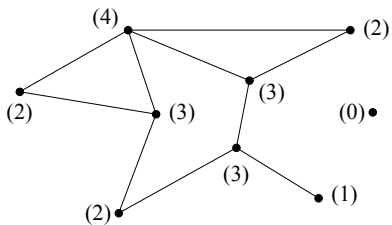
# Outline

- 1 Necessary and sufficient conditions for  $\pi \in NS_n$  to be graphic
- 2 Sufficient conditions for  $\pi \in GS_n$  to be  $A_{r+1}$ -graphic
- 3 Sufficient conditions for  $\pi \in GS_n$  to be almost  $A_{r+1}$ -graphic
- 4 Applications

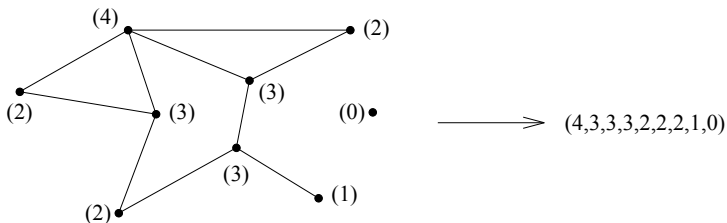
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## Sequences in $NS_n$ and $GS_n$

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 $\pi = (d_1, d_2, \dots, d_n)$  a typical element

$GS_n :=$  All sequences in  $NS_n$  which are degree sequences for some  
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$\sigma(\pi) :=$  The sum of all  $n$  terms of  $\pi$ .

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- It is sufficient that  $\sigma(\pi)$  is even and  $d_{d_1+1} \geq d_1 - 1$  for  $\pi$  to belong to  $GS_n$  (Yin, 2005).

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# A necessary and sufficient condition for $\pi$ to be in $GS_n$ .

Let the threefold process of

- deleting  $d_i$  from  $\pi$ ,
- subtracting 1 from the first  $d_i$  terms of the new sequence, and
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**Theorem (Kleitman and Wang, 1973)**

*A sequence is graphic if and only if it is graphic after laying off any of its terms.*

Note: This generalizes a theorem due to Havel and Hakimi (1955).

# A second characterization of sequences that are graphic.

## Theorem (Erdős and Gallai, 1960)

Let  $\pi = (d_1, d_2, \dots, d_n) \in NS_n$ . Then  $\pi$  is graphic if and only if  $\sigma(\pi)$  is even and, for all  $t \in [n]$ ,

$$\sum_{i=1}^t d_i \leq t(t-1) + \sum_{i=t+1}^n \min(t, d_i).$$

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For example, the sequence  $(6, 6, 5, 4, 3, 2, 2)$  is not graphic since the inequality above does not hold for  $t = 3$ .

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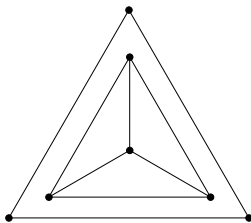
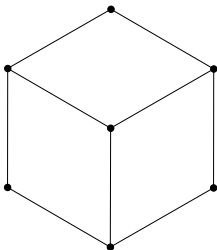
## What is meant by $A_{r+1}$ -graphic?

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Caution: A sequence can be  $A_{r+1}$ -graphic and have realizations that don't even contain a  $(r+1)$ -clique. For example,  $\pi = (3, 3, 3, 3, 2, 2, 2)$  can be realized as each of the following simple graphs.



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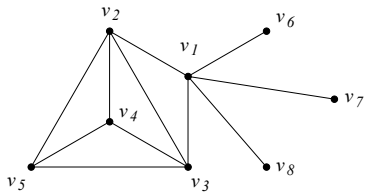
- A sequence  $\pi \in GS_n$  is  $A_{r+1}$ -graphic if a realization of  $\pi$  exists with an  $(r+1)$ -clique on  $(r+1)$ -vertices of maximal degree.
- A sequence  $\pi \in GS_n$  is  $K_{r+1}$ -graphic if a realization of  $\pi$  exists with an  $(r+1)$ -clique.

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- A sequence  $\pi \in GS_n$  is  $K_{r+1}$ -graphic if a realization of  $\pi$  exists with an  $(r+1)$ -clique.
- A sequence is  $A_{r+1}$ -graphic if and only if it is  $K_{r+1}$ -graphic.

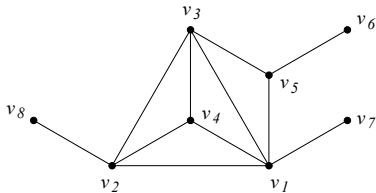
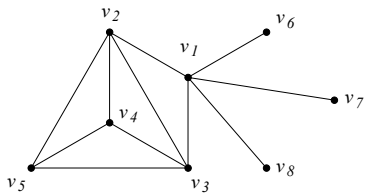
# Example

$(5, 4, 4, 3, 3, 1, 1, 1)$



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### Theorem (Gould, 1999)

*Suppose a sequence  $\pi = (d_1, d_2, \dots, d_n)$  has a realization  $G$  with a subgraph  $H$  on  $r$  vertices. Then there exists a realization  $G^*$  of  $\pi$  with a subgraph  $H^*$  isomorphic to  $H$  such that the vertices of  $H^*$  are the vertices of  $G^*$  corresponding to the first  $r$  terms of  $\pi$ .*

## Sketch of proof

Let  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ .

Let  $G$  be a realization of  $\pi$ .

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Note that  $j < k$  thus  $d_j \geq d_k$ ,  
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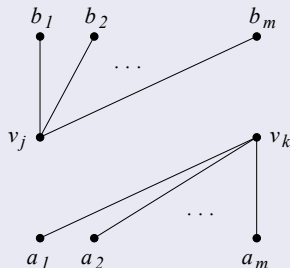
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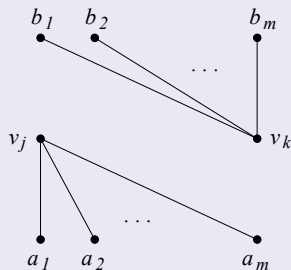
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**Theorem (Rao, 1979)**

*A sequence  $\pi \in NS_n$  is  $A_{r+1}$ -graphic if and only if  $\pi_{r+1}$  is graphic.*

## Two sufficient conditions for $\pi \in GS_n$ to be $A_{r+1}$ -graphic

Let us suppose  $\pi = (d_1, d_2, \dots, d_n) \in GS_n$  where  $n \geq r + 1$ .

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*If  $d_{r+1} \geq r$  and either*

- $d_i \geq 2r - i$  for all  $i \in [r - 1]$ , or*
- $\pi$  has at least  $2r + 2$  terms and  $d_{2r+2} \geq r - 1$*

*then  $\pi$  is  $A_{r+1}$ -graphic.*

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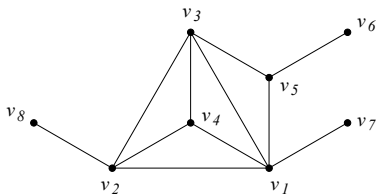
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then  $\pi$  is  $A_{r+1}$ -graphic.

Note this condition is certainly not necessary. For example, we have already learned that  $(5, 4, 4, 3, 3, 1, 1, 1)$  is  $K_4$ -graphic.

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# Almost $A_{r+1}$ -graphic?

We declare  $\pi$  to be almost  $A_{r+1}$ -graphic whenever  $\pi$  has a realization that is one edge shy of containing a copy of  $K_{r+1}$ .

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This condition is not necessary, which can be noted by once again considering the graphic sequence  $(5, 4, 4, 3, 3, 1, 1, 1)$ .

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# A pair of conjectures and recent theorems which can be proven using the tools described in this talk

## Theorem (Conjectured by Erdős in 1991)

Let  $\pi$  be a sequence in  $GS_n$ . Then  $\pi$  is  $A_{r+1}$ -graphic if

- $r^2 \leq \frac{2}{3}n$  and
- $\sigma(\pi) \geq (r-1)(2n-r) + 2$ .

## Theorem (Yin, 2005)

Let  $\pi$  be a sequence in  $GS_n$ . Then  $\pi$  is almost  $A_{r+1}$ -graphic if

- $3r^2 - r - 1 \leq n$  and
- $\sigma(\pi) \geq (r-1)(2n-r) + 2 - (n-r)$ .

# Thank You

Thank you for attending this presentation!

A special thank you also to Dr. John Caughman and Dr. Joyce O'Halloran for helping me prepare this 501 project.