Math 344 - Day 7: Subgroups and Subgroup tests

We saw that the rotations of a square form a group that is contained within the full group of symmetries of a square. This sort of thing happens often.

Definition. Let G be a group. A subset $H \subseteq G$ is called a **subgroup** of G if H itself forms a group with respect to the operation on G. When H is a subgroup of G, we write $H \leq G$.

Using the definition. Prove that the set

$$5\mathbb{Z} := \{a \in \mathbb{Z} \mid \exists n \in \mathbb{Z}\ a = 5n\}$$

is a subgroup of $\mathbb{Z}$.

Proof.

a. *Closure.* Pick any arbitrary elements $a, b \in 5\mathbb{Z}$. Then ... (details) ... So $a + b \in 5\mathbb{Z}$.

b. *Identity.* Recall that 0 is the identity element in $\mathbb{Z}$. Then ... (details) ... So $0 \in 5\mathbb{Z}$.

c. *Inverses.* Let a be any arbitrary element in $5\mathbb{Z}$. Then $-a$ is the inverse element of a in $\mathbb{Z}$. Then ... (details) ... So $-a \in 5\mathbb{Z}$.

d. *Associativity.* Let a, b, c be any arbitrary elements in $5\mathbb{Z}$. Then ... (details) ... So $(a + b) + c = a + (b + c)$.

Discussion. You might notice that the tests for identity and inverses are stated a bit differently above than they were in the axioms for a group. How are the statements different and why are they equivalent?

Discussion. Let's see if we can find a smaller, more efficient way to test a subset to see if it is a subgroup.

Subgroup Test One. Let H be a non-empty subset of a group G such that:

1. For all $a, b \in H$, we have $ab \in H$, and
2. For all $a \in H$, we have $a^{-1} \in H$.

Then H is a subgroup of G.

Subgroup Test Two. Let H be a non-empty subset of a group G such that:

1. $|H|$ is finite, and
2. For all $a, b \in H$, we have $ab \in H$.

Then H is a subgroup of G.

Subgroup Test Three. Let H be a non-empty subset of a group G such that:

1. For all $a, b \in H$, we have $ab^{-1} \in H$.

Then H is a subgroup of G.

Discussion. Why do these tests work?
