An Introduction to Distance-Regular Graphs

Groups, Graphs and Algebras

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Groups

A *binary operation on a set* G is a mapping: $* : G \times G \to G$. For $(a, b) \in G \times G$, we write $a * b$, or ab, for '$*((a, b))$'.

Definition 1 A pair $(G, *)$ consisting of a set G and a binary operation $*$ on G is said to be a *group* if it satisfies the following.

(i) *Associative law*: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.

(ii) *Existence of identity*: There is an element $e \in G$, called *the identity element* and often denoted by 1, such that $a * e = a = e * a$ for all $a \in G$.

(iii) *Existence of inverse of each element*: Each element $a \in G$ has an element $a' \in G$, called *the inverse* of a and often denoted by a^{-1}, such that $a * a' = e = a' * a$.

A group is called *finite*, or a *finite group*, if G is a finite set.

Uniqueness of Identity Element: Let $(G, *)$ be a group and $e, \hat{e} \in G$.

$$a * e = a = e * a \text{ and } a * \hat{e} = a = \hat{e} * a \text{ for all } a \in G \Longrightarrow e = \hat{e}.$$

Proof. $\hat{e} = e * \hat{e} = e$. ∎

Subgroups, Normal Subgroups and Simple Groups

Let $(G, *)$ be a group.

(i) For subsets A, B of G, $A * B = \{a * b \mid a \in A, b \in B\}$, we write $a * B$ for $\{a\} * B$. This defines a binary operation on $\mathcal{P}(G)$, the power set of G.

(ii) A subset H of G is called a *subgroup* if $a * b \in H$ for all $a, b \in H$ and H becomes a group with respect to the binary operation $*$.

(iii) A subgroup H of G is said to be a *normal subgroup* if $g^{-1}Hg = H$ for all $g \in G$. We wirte $H \lhd G$ in this case.

(iv) If $H \lhd G$, then $\{a * H \mid a \in G\}$ becomes a group with respect to the binary operation defined on $\mathcal{P}(G)$ in (i). This group is denoted by G/H.

(v) G is said to be a *simple group* if $G \neq \{1\}$ and that $H \lhd G$ implies $H = \{1\}$ or G.

(vi) If G is a finite group, then there is a series of subgroups $H_0, H_1, \ldots, H_n$ of G such that

$$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G, \text{ and that } H_i/H_{i-1} \text{ is a simple group for } i = 1, 2, \ldots, n.$$

Classification Theorem of Finite Simple Groups

Definition 2 Two groups $(G, *)$ and $(H, \circ)$ are said to be *isomorphic* whenever there is a bijection $\phi : G \to H$ such that $\phi(a * b) = \phi(a) \circ \phi(b)$ for all $a, b \in G$.

Theorem 1 *Let G be a finite simple group. Then G is isomorphic to one of the following groups:*

Cyclic groups of prime order: $\mathbf{Z}_p$.

Alternating groups: A_n.

Classical type groups:

$PSL_n(q)$, $PSp_n(q)$, $P\Omega^+_{2n}(q)$, $P\Omega^-_{2n}(q)$, $P\Omega_{2n+1}(q)$, $PSU_n(q)$ *(Classical groups).*
$G_2(q)$, $F_4(q)$, $F_4(2)'$, $E_6(q)$, $E_7(q)$, $E_8(q)$ *(Chevalley groups).*
$Sz(q)$, $Re(q)$, ${}^2E_6(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$ *(Steinberg type groups).*

26 sporadic groups: M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, J_4, HS, Suz, M^c, Ru, He, Ly, ON, Co_1, Co_2, Co_3, $M(22)$, $M(23)$, $M(24)'$, F_2, F_3, F_5, F_1

M. Ronan, *Symmetry and the Monster: The Story of One of the Greatest Quests of Mathematics*, Oxford University Press, USA (July 1, 2006). — for general reader

Proof of the Classification Theorem

1. Determination of the list of known finite simple groups.

   The set of finite simple groups in this list is denoted by $\mathcal{K}$.

2. Characterization of finite simple groups of classical type as groups having '*split BN-pair of rank at least three*' and its classification. — By J. Tits.

3. Classification of 'small' finite simple groups in various sense.

4. Big induction by local analysis.

   (a) *Induction* by taking a minimum counter example G:
   G is a finite simple group not in $\mathcal{K}$ such that for each subgroup H of G and $N \lhd H$ with $H \neq G$, if H/N is a simple group then $H/N \in \mathcal{K}$.

   (b) *Local Analysis* by considering the structure of subgroups

   $$N_G(P) := \{g \in G \mid g^{-1}Pg = P\}$$

   for various subgroups $P \neq \{1\}$ of G with prime power number of elements.

   (c) Deduce a contradiction to show there is no such G.

Current Proof, the First Generation Proof

1. Too long. Over 20,000 journal pages, by many mathematicians.

   Identification of finite simple groups of certain properties naturally require long proof and 1000 page proof seems impossible.

2. Not enjoyable to argue in a big induction to show a contradiction.

Consequences of the Current Proof

1. No one has read the proof from the beginning to the end by oneself.

2. Difficult to encourage promising young mathematicians to do its revision.

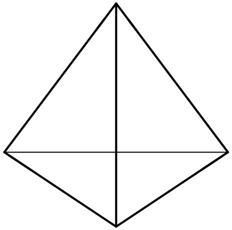
Consequences of the Classification Theorem

1. Set foundation to investigate the structure of general finite groups.

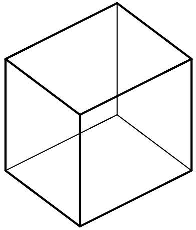
2. There are many applications, and many open problems on finite groups were solved.

3. Opened up rich fields of mathematics in connection with finite geometry, coding theory, mysteries connected to the Monster simple group.

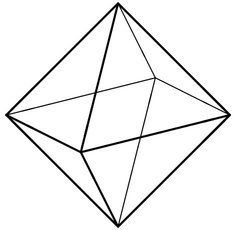
What's next?



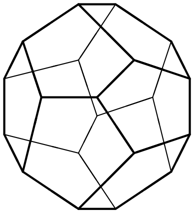
Tetrahedron



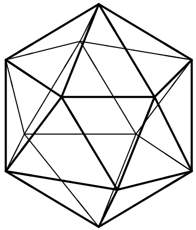
Cube



Octahedron



Dodecahedron



Icosahedron

Graphs

$\Gamma = (X, R)$: a Graph

- X: a nonempty finite set, called the vertex set of Γ
- $R \subset \{\{x, y\} \mid x, y \in X\}$: edge set
- When $\{x, y\} \in R$, write $x \sim y$ and say x is adjacent to y.
- $(x_0, x_1, \ldots, x_\ell)$ is a path connecting x_0 and x_ℓ of lenght ℓ if $x_i \in X$ and $x_0 \sim x_1 \sim \cdots \sim x_\ell$.
- Γ is connected if for every $x, y \in X$, there is a path connecting x and y.
- $\partial(x, y)$: the length of a shortest path between x and y, and is called distance between x and y. If there is no path between x and y, then $\partial(x, y) = \infty$.
- $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$: the set of vertices at distance i from x.
- For $Y \subset X$, $\Gamma_i(Y) = \{z \in X \mid \min\{\partial(z, y) \mid y \in Y\} = i\}$.

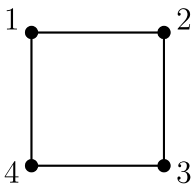
Adjacency Matrix of a Graph

Let $\Gamma = (X, R)$ be a graph.

The adjacency matrix $A \in \mathrm{Mat}_X(\boldsymbol{C})$ of Γ is a matrix whose rows and columns are indexed by X such that

$$(A)_{x,y} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1 [Square C] $X = \{1, 2, 3, 4\}$, $R = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$.



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Spectrum of a Graph

Let A be the adjacency matrix of a graph $\Gamma = (X, R)$.

Since A is a symmetric $(0, 1)$ matrix, all eigenvalues are real and A is diagonalizable.

Let $\theta_0 > \theta_1 > \cdots > \theta_s$ be distinct eigenvalues of A with multiplicities $m_0, m_1, \ldots, m_s$. Then the following is called the spectrum of Γ (or A).

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_s \\ m_0 & m_1 & \cdots & m_s \end{pmatrix}.$$

The characteristic polynomial $\det(\lambda I - A)$ of A is also called the characteristic polynomial of Γ and written as $\mathrm{Char}_\Gamma(t)$. Then

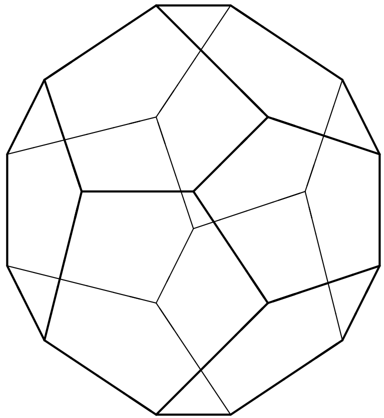
$$\mathrm{Char}_\Gamma(t) = (t - \theta_0)^{m_0}(t - \theta_1)^{m_1} \cdots (t - \theta_s)^{m_s}.$$

The spectrum does not depend on the ordering of vertices.

Example 2 For the square graph in Example 1, $\mathrm{Char}_C(t) = (t - 2)t^2(t + 2) = t^4 - 4t^2$,

$$\mathrm{Spec}(C) = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Do you want to determine the eigenvalues of the Dodecahedron?



Dodecahedron

Let $\Gamma = (X, R)$ be the dodecahedron graph. Then $|X| = 20$ and $|R| = 30$. So the adjacency matrix A is a 20×20 matrix.

Distance-Regular Graphs (DRG)

$\Gamma = (X, R)$: a connected graph with vertex set X and edge set R.

- $\partial(x, y)$: the distance between x and y.
- $d(\Gamma) = \max\{\partial(x, y) \mid x, y \in X\}$: the diameter of Γ.
- For $u \in X$ and $j \in \{0, 1, \ldots, d(\Gamma)\}$, let

$$\Gamma_j(u) = \{x \in X \mid \partial(u, x) = j\},\ \Gamma(u) = \Gamma_1(u).$$

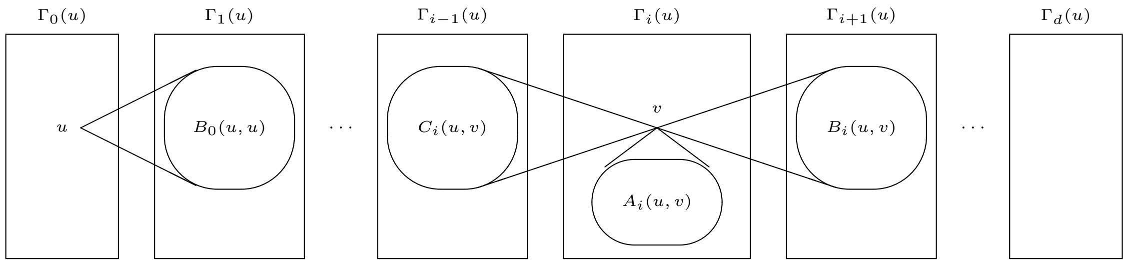
Definition 3 A connected graph $\Gamma = (X, R)$ is said to be *distance-regular* (DR), or a *distance-regular graph* (DRG), if the following numbers

$$\begin{aligned} c_i &= |\Gamma_{i-1}(u) \cap \Gamma(v)|, \\ a_i &= |\Gamma_i(u) \cap \Gamma(v)|, \text{ and} \\ b_i &= |\Gamma_{i+1}(u) \cap \Gamma(v)| \end{aligned}$$

depend only on $i = \partial(u, v)$ for all $i \in \{0, 1, 2, \ldots, d(\Gamma)\}$.

- For $u, v \in X$ with $\partial(u, v) = j$ let

$$
\begin{aligned}
C(u,v) &= C_j(u,v) = \Gamma_{j-1}(u) \cap \Gamma(v), \\
A(u,v) &= A_j(u,v) = \Gamma_j(u) \cap \Gamma(v), \text{ and} \\
B(u,v) &= B_j(u,v) = \Gamma_{j+1}(u) \cap \Gamma(v).
\end{aligned}
$$

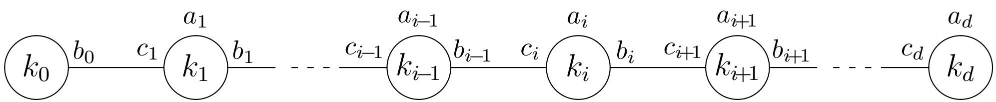


- DRG is a regular graph of valency $k = b_0$, and $k = b_i + a_i + c_i$ for all $0 \leq i \leq d$.

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\} = \left\{ \begin{array}{cccccc} * & c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_{d-1} & * \end{array} \right\}$$

  is called the *intersection array* of Γ.

We often draw the following diagram to show parameters, where $k_i = |\Gamma_i(x)|$.

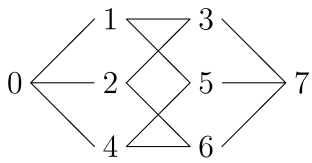


Cube

Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and

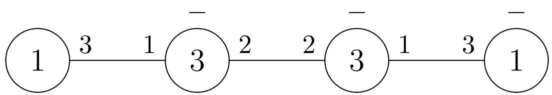
$$R = \{\{0, 1\}, \{0, 2\}, \{0, 4\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 6\}, \{4, 5\}, \{4, 6\}, \{3, 7\}, \{5, 7\}, \{6, 7\}\}.$$

This is a cube and is distance-regular of diameter 3 and valency 3. The intersection array is given below.



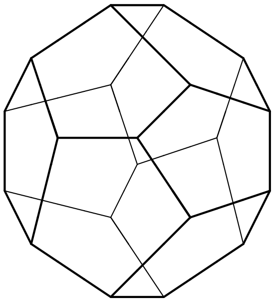
$$\left\{\begin{matrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{matrix}\right\}.$$

$$\Gamma_0(0) = \{0\},\ \Gamma_1(0) = \{1, 2, 4\},\ \Gamma_2(0) = \{3, 5, 6\} \text{ and } \Gamma_3(0) = \{7\}.$$



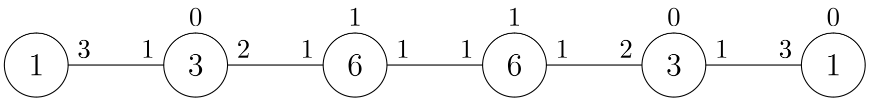
Dodecahedron

$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}.$



The dodecahedron is a distance-regular graph of diameter 5 and valency 3.

$$\left\{ \begin{array}{cccccc} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{array} \right\}.$$



Bose-Mesner Algebra

Let $\Gamma = (X, R)$ be a distance-regular graph of valency k and diameter d. Let $A_i \in \mathrm{Mat}_X(\boldsymbol{C})$ be its i-th adjacency matrix of Γ define by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \text{ for } i = 0, 1, \ldots, d \text{ with } A_{-1} = A_{d+1} = O$$

Let $v_0(t) = 1$, $v_1(t) = t$ and define $v_{i+1}(t)$ by the following relation: For $i = 1, 2, \ldots, d$

$$t \cdot v_i(t) = b_{i-1}v_{i-1}(t) + a_iv_i(t) + c_{i+1}v_{i+1}(t) \text{ for } i = 1, 2, \ldots, d \text{ with } c_{d+1} = 1.$$

Then $v_i(A) = A_i$ and $v_{d+1}(A) = 0$.

The following algebra is called the *Bose-Mesner algebra* of Γ.

$$\mathcal{M} = \{v(A) \mid v(t) \in \boldsymbol{R}[t]\}.$$

By the definition above $A_0 = I, A_1 = A, \ldots, A_d \in \mathcal{M}$.

Bose-Mesner Algebra and Bigg's Formula

Theorem 2 (N. Biggs) *Let* $\Gamma = (X, R)$ *be a distance-regular graph of valency* k *and diameter* d*. Then the following hold.*

(i) $\mathcal{M} = \mathrm{Span}(I, A, A^2, \ldots, A^d) = \mathrm{Span}(A_0, A_1, A_2, \ldots, A_d)$.

(ii) $v_{d+1}(t)$ *is a minimal polynomial of* A *and the set of distinct eigenvalues of* A *and the roots of* $v_{d+1}(t)$ *coincide.*

(iii) *Let* $\theta_0 > \theta_1 > \ldots > \theta_d$ *be distinct eigenvalues of* A*. Then* $k = \theta_0$ *and*

$$m(\theta_j) = \frac{|X|}{\displaystyle\sum_{i=0}^{d} \frac{v_i(\theta_j)^2}{v_i(\theta_0)}} \quad \text{for } j = 0, 1, \ldots, d.$$

Example 3 [Dodecahedron] $v_0(t) = 1,\ v_1(t) = t,\ v_2(t) = t^2 - 3,\ v_3(t) = t^3 - t^2 - 5t + 3,$

$v_4(t) = \frac{1}{2}(t^4 - 2t^3 - 5t^2 + 8t), v_5(t) = \frac{1}{6}(t^5 - 2t^4 - 7t^3 + 10t^2 + 10t - 6), v_6(t) = \frac{1}{6}t(t-3)(t-1)(t+2)(t^2-5)$

$$\mathrm{Spec}(C) = \begin{pmatrix} 3 & \sqrt{5} & 1 & 0 & -2 & -\sqrt{5} \\ 1 & 3 & 5 & 4 & 4 & 3 \end{pmatrix}.$$

List of Known Primitive Distance-Regular Graphs

A distance-regular graph $\Gamma = (X, R)$ is said to be *primitive* if the graph $(X, R^{(i)})$ is connected for all $i = 1, 2, \ldots, d$, where $R^{(i)} = \{(x, y) \in X \times X \mid \partial(x, y) = i\}$.

Cycles of Prime Length: C_p.

Root System Type: $J(n, d)$, $H(d, n)$ and their related graphs.

Classical Type: $J_q(n, d)$, $H_q(d, n)$, Dual polar graphs, quadratic forms graphs, alternating forms graphs, Hermitian forms graphs and their related graphs.

Sporadic Type: $GO(s, t)$, etc.

Diameter 2 and 3:

- Cycles of prime length, root system type and classical type satisfy an extra condition called the *Q-polynomial property*. A distance-regular graph with this property is called a *Q-distance-regular graph*.

Project Distance-Regular Graph

Classify (parameters of) primitive distance-regular graphs with 'large' diameter.

1. Collect (parameters of) all known primitive distance-regular graphs.

2. Determine whether or not there is a series of primitive distance-regular graphs with unbounded diameter which are not Q-polynomial.

3. Classify (parameters of) primitive Q-distance-regular graphs.

4. Develop representation theory of distance-regular graphs.

5. Develop techniques to investigate structures of distance-regular graphs.

The problem was first proposed by E. Bannai in 1980's.

Current Status

1. E. Bannai made the first list of known primitive distance-regular graphs.
2. No infinite series of primitive non-Q-polynomial distance-regular graphs with unbounded diameter is found yet.
3. Q-distance-regular graphs:
    - Parameters of Q-distance-regular graphs were shown to be written by 5 parameters. — by D. Leonard.
    - Parameters of Q-distance-regular graphs were expressed by 5 parameters in closed form and divided into several classes. — by E. Bannai and T. Ito.
    - Some classes of Q-distance-regular graphs were classified. — by P. Terwilliger, G. Dickie, C-W. Weng, J. Caughman and others.
4. Representation theory:

   Irreducible modules of the subsonstituent (or Terwilliger) algebra of Q-distance-regular graphs were classified. — by Terwilliger, T. Ito, K. Nomura and others.
5. Structure theory: Developed by many.

Completely Regular Codes (CRC)

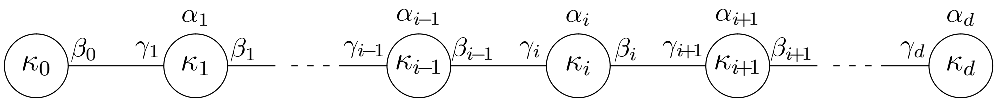
Definition 4 Let $\Gamma = (X, R)$ be a connected graph. A subset C of X is said to be *completely regular* (CR), or a *completely regular code* (CRC), if the following numbers

$$\begin{aligned}\gamma_i &= |\Gamma_{i-1}(C) \cap \Gamma(u)|,\\ \alpha_i &= |\Gamma_i(C) \cap \Gamma(u)|, \text{ and}\\ \beta_i &= |\Gamma_{i+1}(C) \cap \Gamma(u)|.\end{aligned}$$

depend only on $i = \partial(u, C)$ for all $i \in \{0, 1, 2, \ldots, t(C)\}$.

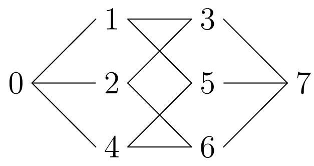
This definition is equivalent to the condition that the principal module of $\mathcal{T}(\Gamma; C)$ is thin irreducible.

We often draw the following diagrams to show parameters, where $\kappa_i = |\Gamma_i(C)|$.

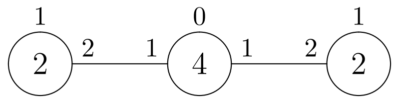


Completely Regular Codes in Cube

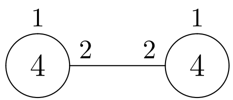
Since the cube is distance regular, every singleton is completely regular. The following subsets are also completely regular.



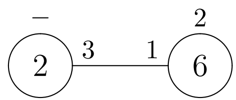
- $\{0, 1\}$: edge



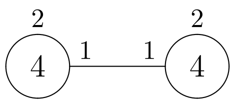
- $\{0, 1, 6, 7\}$: two edges



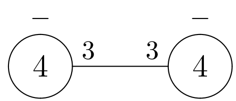
- $\{0, 7\}$: antipodes



- $\{0, 1, 5, 4\}$: quadrangle



- $\{0, 3, 5, 6\}$: mutual distance 2



Problem

All known distance-regular graphs have many completely regular codes.

Classify distance-regular graphs having ‘good’ completely regular codes.

Classification

Quadrangle, i.e. Square

Theorem 3 (2008) *Let* $\Gamma = (X, R)$ *be a distance-regular graph of diameter* $d \geq 4$. *Suppose there is a quadrangle, and every quadrangle is a completely regular code. Then one of the following holds.*

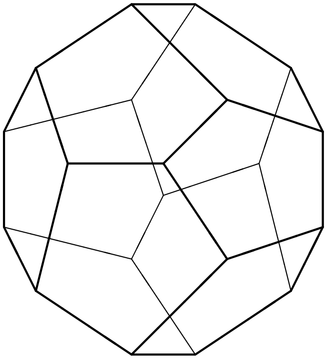
(i) *The binary Hamming graph* $H(d, 2)$.

(ii) *The folded graph of a binary Hamming graph* $H(2d + 1, 2)$.

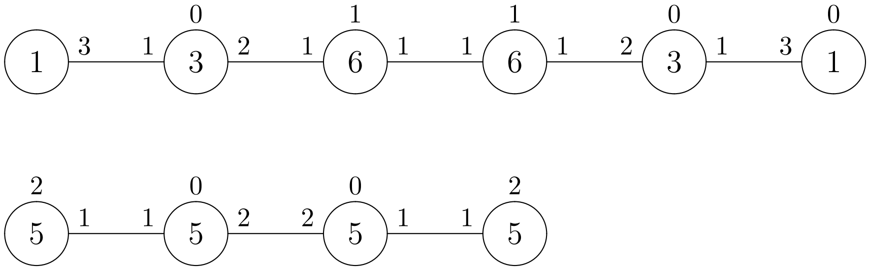
(iii) *The folded graph of a binary Hamming graph* $H(2d, 2)$.

(iv) *The coset graph of the extended binary Golay code with intersection array:*
$\{24, 23, 22, 21; 1, 2, 3, 24\}$.

Pentagon



Dodecahedron



Proposition 4 (2008) *Let* Γ *be a distance-regular graph of diameter* $d \geq 3$. *If a pentagon in* Γ *is a completely regular code with covering radius* $d-2$, *then* Γ *is isomorphic to the Dodecahedron.*

Tridiagonal Space of a Graph w.r.t a Subset C

$\Gamma = (X, R)$: a connected (finite simple) graph with path-distance $\partial(x, y)$.

- A: adjacency matrix of Γ in $\mathrm{Mat}_X(\boldsymbol{C})$.
- $V = \boldsymbol{C}^X = \mathrm{Span}(\hat{x} \mid x \in X)$, where $\hat{x}$ is the unit vector with 1 at x-entry.
- $\emptyset \neq C \subset X$, $C_i = \Gamma_i(C) = \{x \in X \mid \min\{\partial(x, y) \mid y \in C\} = i\}$ and

$$V_i^* = \mathrm{Span}(\hat{x} \mid x \in C_i).$$

- $t = \max\{i \mid C_i \neq \emptyset\} = \max\{\partial(x, y) \mid x \in X, \text{ and } y \in C\}$ (covering radius).

Then for $i \in \{0, 1, \ldots, t\}$ with $V_{-1}^* = V_{t+1}^* = 0$

$$AV_i^* \subset V_{i-1}^* + V_i^* + V_{i+1}^* \text{ for all } i \in \{0, 1, \ldots, t\},$$

and $\Phi = (A, V; V_0^*, V_1^*, \ldots, V_t^*)$ is a tridiagonal space.

The Terwilliger algebra $\mathcal{T}(\Phi)$ in this case is written as $\mathcal{T}(\Gamma; C)$.

THANK YOU!!