Mth 322: Midterm Exam Solutions

Problem 1 (10 points). Given the partial differential equation (p.d.e.) for $z(x, y)$

$$yz_x + xz_y = z + 1 \tag{1}$$

(a)6 points: Find two functionally independent first integrals u_1 and u_2

(b)2 points: Write the expression of the general integral to the p.d.e.

(c)2 points: Provide the explicit form of two particular solutions $z_1(x, y)$ and $z_2(x, y)$ to the p.d.e.

Solution

(a) The system of equations associated to (1) is

$$\frac{d\,x}{y} = \frac{d\,y}{x} = \frac{d\,z}{z+1} = \frac{d\,(x+y)}{(x+y)}$$

where the last equality is obtained by adding the first two ratios. Then

$$xdx = ydy \Rightarrow x^2 - y^2 = c_1 \Rightarrow u_1 = x^2 - y^2$$

and

$$\frac{d\,z}{z+1} = \frac{d\,(x+y)}{(x+y)} \Rightarrow \ln|z+1| = \ln|x+y| + c_2 \Rightarrow \ln\left|\frac{z+1}{x+y}\right| = c_2 \Rightarrow u_2 = \frac{z+1}{x+y}$$

(b) The general integral is

$$F\left(x^2 - y^2, \frac{z+1}{x+y}\right) = 0$$

or, after solving for z,

$$z = (x+y)f(x^2 - y^2) - 1$$

where f is an arbitrary continuously differentiable function.

(c)

$$f \equiv 0 \Rightarrow z(x, y) \equiv -1$$

$$f \equiv 1 \Rightarrow z(x, y) = x + y - 1$$

Problem 2 (10 points). Find the solution $z(x, y)$ to the p.d.e.

$$xz_x + yz_y = z - 1 \tag{2}$$

$$\tag{3}$$

that passes through the curve $\mathcal{C}$ of parametric equations

$$x_0(t) = t, \quad y_0(t) = t^2, \quad z_0(t) = 2t, \quad t > 0 \tag{4}$$

Solution

The system of equations associated to (2) is

$$\frac{d\,x}{x} = \frac{d\,y}{y} = \frac{d\,z}{z - 1}$$

which gives the first integrals

$$u_1 = \frac{x}{y}, \quad u_2 = \frac{z - 1}{x}$$

Along the curve $\mathcal{C}$,

$$u_1 = \frac{t}{t^2} = \frac{1}{t}$$

$$u_2 = \frac{2t - 1}{t} = 2 - \frac{1}{t} = 2 - u_1$$

The solution to the initial value problem is thus

$$u_1 + u_2 = 2 \Rightarrow \frac{x}{y} + \frac{z - 1}{x} = 2 \Rightarrow z = 1 + 2x - \frac{x^2}{y}$$

Problem 3 (10 points). Find the solution to the initial value problem

$$z_x + z_y + 2z = 0 \tag{5}$$
$$z(x,0) = \sin(x), \quad -\infty < x < \infty \tag{6}$$

Solution: From

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{-2z}$$

we obtain

$$x - y = c_1 \Rightarrow u_1 = x - y$$

$$\ln|z| = -2x + c_2 \Rightarrow z = c_2 e^{-2x} \Rightarrow u_2 = ze^{2x}$$

The general integral is thus

$$F(x-y, ze^{2x}) = 0 \Rightarrow z = e^{-2x} f(x-y)$$

By imposing the initial condition (6),

$$z(x,0) = e^{-2x} f(x) = \sin(x) \Rightarrow f(x) = e^{2x}\sin(x)$$

Then

$$z = e^{-2x} e^{2(x-y)} \sin(x-y) = e^{-2y}\sin(x-y)$$

Problem 4 (10 points). Consider the p.d.e.

$$y^2 z_x + 2xz_y = z \tag{7}$$

(a) 6 points: Given the initial condition

$$x = t, \quad y = 2t, \quad z = t^2, \quad 0 < t < \infty \tag{8}$$

and the point $P(1,2,1)$ on this curve, which of the following statements is true in a vicinity of P:

1. There is only one solution to the problem (7-8).
2. There is an infinite number of solutions to the problem (7-8).
3. There is no solution to the problem (7-8).

Justify your answer.

(b) 4 points: Find the coordinates of a point Q on the curve (8) such that in a vicinity of Q there is no solution to the problem (7-8).

Solution:
(a) $P(1,2,1)$ corresponds to $t = 1$. Also, from (8)

$$x' = 1, \quad y' = 2, \quad z' = 2t$$

and the coefficients of the p.d.e. are

$$\mathcal{P} = y^2 = 4t^2, \quad \mathcal{Q} = 2x = 2t, \quad \mathcal{R} = z = t^2$$

Solvability condition at $t = 1$

$$\mathcal{Q}x' - \mathcal{P}y' = 2 - 8 = -6 \neq 0$$

Statement (1) is thus correct: There is only one solution to the problem (7-8) in a vicinity of $P(1,2,1)$

(b) In the vicinity of a point on $\mathcal{C}$ defined by the parameter value $t_0 > 0$, $Q(x(t_0), y(t_0), z(t_0))$, there will be no solution to the IVP if

$$\frac{x'(t_0)}{\mathcal{P}(t_0)} = \frac{y'(t_0)}{\mathcal{Q}(t_0)} \neq \frac{z'(t_0)}{\mathcal{R}(t_0)}$$

or, using the relations above,

$$\frac{1}{4t_0^2} = \frac{1}{t_0} \neq \frac{2}{t_0}$$

From the first two ratios we get $t_0 = 1/4$ (notice $t_0 > 0$) and at $t_0 = 1/4$,

$$4 = 4 \neq 8$$

The point Q is thus the point of coordinates

$$Q\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{16}\right)$$