Homework #3 - due 11/25/2009

Hand in solutions to the following problems:

P1 (15 points): Let $\Omega = \{x \in R^n : |x| < 1\}$, where $|\cdot|$ denotes the euclidian norm in R^n (Ω is the open unit ball in R^n). Consider the function $u(x) = |x|^\alpha$.

(5p) For $n = 1$, find the values of α such that $u \in H^1(\Omega)$.

(10p) For $n = 2$, find the values of α such that $u \in H^1(\Omega)$.

P2 (15 points) Consider the following problem in a bounded domain $\Omega \subset R^2$:

$$\begin{aligned} -\Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u &= f, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega \end{aligned}$$

where β_1, β_2 are constants.

(5p) Give a variational formulation to this problem.

(10p) Show that the variational problem has an unique solution for any values of the constants β_1, β_2.

P3 (15 points): For a bounded domain $\Omega \subset R^n$, let

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx}$$

(5p) Prove that $\lambda_1 > 0$.

(10p) Prove that, for every $f \in L^2(\Omega)$ and for any constant $c > -\lambda_1$, the Dirichlet problem

$$\begin{aligned} -\Delta u + cu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has an unique weak solution $u \in H_0^1(\Omega)$.

P4 (15 points): Let $\Omega = (0, 1) \times (0, 1)$. Show that

$$\int_\Omega v^2 \, dx \, dy \le \int_\Omega |\nabla v|^2 \, dx \, dy, \quad \forall v \in H_0^1(\Omega)$$