Problem 1 (50 points) Consider the wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0, \quad x \in \Omega, \quad t > 0 && (1)\\ u &= 0, \quad \text{on } \partial\Omega \times (0,\infty) && (2)\\ u(x,0) &= u_0(x) \quad x \in \Omega && (3)\\ u_t(x,0) &= v_0(x) \quad x \in \Omega && (4) \end{aligned}$$

Introduce a new variable $v = u_t$ to obtain an equivalent formulation

$$\begin{aligned} u_t - v &= 0 && (5)\\ v_t - \Delta u &= 0 && (6)\\ u(x,0) &= u_0(x) && (7)\\ v(x,0) &= v_0(x) && (8)\\ u|_{\partial\Omega} &= 0 && (9) \end{aligned}$$

For $t \geq 0$ we view

$$U = \begin{bmatrix} u \\ v \end{bmatrix}$$

as a function of t with values in an appropriate Hilbert space. Consider the Hilbert space[1]

$$H = H_0^1(\Omega) \times L^2(\Omega)$$

with the inner product

$$(U_1, U_2)_H = \int_\Omega \nabla u_1 \nabla u_2 \, dx + \int_\Omega u_1 u_2 \, dx + \int_\Omega v_1 v_2 \, dx$$

where

$$U_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

Define the linear operator

$$A : D(A) \subset H \to H, \quad AU = \begin{bmatrix} -v \\ -\Delta u \end{bmatrix}$$

where $D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. Given $U_0 \in D(A)$, we are interested to study the existence of the solution of the IVP

$$\begin{aligned} \frac{dU}{dt} + AU &= 0 && (10)\\ U(0) &= U_0 && (11) \end{aligned}$$

[1]notice that the boundary conditions are included in the definition of H

(10 points) Show that $A + I : D(A) \to H$ is monotone[2]

(20 points) Show that $A + I : D(A) \to H$ is maximal monotone

Given $U_0 \in D(A)$, apply the Hille-Yosida theorem to conclude that there is an unique solution to

$$\frac{d\tilde{U}}{dt} + A\tilde{U} + \tilde{U} = 0 \quad (12)$$

$$\tilde{U}(0) = U_0 \quad (13)$$

(10 points) Show that $U(t) = e^t \tilde{U}(t)$ solves (10-11).

(10 points) The Hille-Yosida theorem states that the solution $U(t)$ has regularity

$$U \in C([0,\infty); D(A)) \cap C^1([0,\infty); H)$$

Using this result, what is the regularity of the solution $u(t)$?

(fill in the dots $u \in C([0,\infty);\ldots) \cap \ldots([0,\infty);\ldots) \cap \ldots([0,\infty);\ldots)$).

Problem 2 (20 points)[3] Show that there is at most one smooth solution to the initial boundary value problem (telegraph equation)

$$u_{tt} + du_t - u_{xx} = f, \quad (x,t) \in (0,1) \times (0,T) \quad (14)$$

$$u = 0, \quad (x,t) \in \{0,1\} \times (0,T) \quad (15)$$

$$u = g,\ u_t = h, \quad (x,t) \in (0,1) \times \{0\} \quad (16)$$

where d is a constant.

Problem 3 (30 points) Write Problem 2 above in the form

$$\frac{dU}{dt} + AU = F \quad (17)$$

$$U(0) = U_0 \quad (18)$$

and give a result of existence of the solution.

[2] $(A + I)U = AU + U$

[3] From Evans, section 7.5, problem 9