Convergence

Let $X_1, X_2, \ldots$ be a sequence of random variables. The sequence $\{X_n\}$ converges in probability to the random variable X

if $\forall \varepsilon > 0, \lim_{n\to\infty} P[|X_n - X| \geq \varepsilon] = 0$

(equivalently, $\forall \varepsilon > 0$ $\lim_{n\to\infty} P[|X_n - X| < \varepsilon] = 1$)

②

Write $X_n \xrightarrow{P} X$

Suppose X is "degenerate", i.e. $P(X = c) = 1$, then we write $X_n \xrightarrow{P} c$

Suppose that $\lim_{n\to\infty} E[X_n] = a$ and $\lim_{n\to\infty} V[X_n] = 0$

Then $P[|X_n - a| \geq \varepsilon]$

$$= P[|X_n - E(X_n) + E(X_n) - a| \geq \varepsilon]$$

③

$$\leq P\left[\,|X_n - E(X_n)| + |E(X_n) - a| \geq \varepsilon\,\right]$$

$$= P\left[\,|X_n - E(X_n)| \geq \varepsilon - |E(X_n) - a|\,\right]$$

By 1st assumption, there is a suff. large n that

$$\varepsilon - |E(X_n) - a| > 0$$

$$\leq \frac{Var(X_n)}{(\varepsilon - |E(X_n) - a|)^2} \quad \text{by Chebyshev}$$

④

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} P[|X_n - a| \geq \varepsilon] \leq \lim_{n\to\infty} \frac{Var(X_n)}{c^2} = 0$$

by our 2nd assumption

$$\therefore X_n \xrightarrow{P} a$$

Let $X_1, \ldots, X_n, \ldots$ be iid with mean μ and var. σ^2.

Consider $\bar{X}_1, \bar{X}_2, \ldots$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

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$$E[\bar{X}_n] = \mu \quad \text{and} \quad V[\bar{X}_n] = \frac{\sigma^2}{n}$$

$$\lim_{n\to\infty} E[\bar{X}_n] = \mu \quad \text{and} \quad \lim_{n\to\infty} V[\bar{X}_n] = 0$$ } called "2nd order conv."

$$\therefore \quad \bar{X}_n \xrightarrow{P} \mu$$

This result is called the Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be iid $N(\mu, \sigma^2)$

Let $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

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$$E[S_n^2] = \sigma^2 \qquad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$V[\,\cdot\,] = 2(n-1)$$

$$\frac{(n-1)^2}{\sigma^4} V[S^2] = 2(n-1)$$

$$V[S^2] = \frac{2\sigma^4}{(n-1)}$$

$$\therefore \quad S_n^2 \xrightarrow{P} \sigma^2$$

Defn: If T_n is an estimator of θ and $T_n \xrightarrow{P} \theta$, then T_n is a *consistent estimator of θ*.

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Thm: Let $X_n \xrightarrow{P} X$ and $h(x)$ is a continuous function.

Then $h(X_n) \xrightarrow{P} h(X)$ [Exercise 5.39]

This implies that $S_n \xrightarrow{P} \sigma$ (even though S_n is biased)

Defn: $X_1, X_2, \ldots$ converges almost surely to X if $\forall \varepsilon > 0$, $P\left(\lim_{n\to\infty} |X_n - X| < \varepsilon\right) = 1$

Write $X_n \xrightarrow{a.s.} X$

(8)

Thm: $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$

(proof in the 600-level course)

The SLLN says $\bar{X}_n \xrightarrow{a.s.} \mu$

(Recall WLLN says $\bar{X}_n \xrightarrow{P} \mu$)

Defn: Let $F_n(x)$ be the cdf of X_n.

If $F(x)$ is a valid cdf and if

$\lim_{n\to\infty} F_n(x) = F(x)$ $\forall x$ such that $F(x)$ is continuous

(9)

then $F(x)$ is called the limiting distribution of X_n and write $X_n \xrightarrow{\mathcal{D}} X$

Example: let $X_1, X_2, \ldots$ be iid $Unif(0,\theta)$

$$f(x_i) = \frac{1}{\theta}, \quad 0 < x_i < \theta$$

let $Y_n = \max\{X_1, \ldots, X_n\}$

let $Z_n = n(\theta - Y_n)$

Find the limiting distribution of Z_n.

(10)

$$G_n(y) = P[Y_n \le y]$$
$$= P[\max(X_1, \ldots, X_n) \le y]$$
$$= (P[X_1 \le y])^n$$
$$= \begin{cases} (y/\theta)^n & 0 < y < \theta \\ 0 & y < 0 \\ 1 & y > \theta \end{cases}$$

$$H_n(z) = P(Z_n \le z) = P(n(\theta - Y_n) \le z)$$

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$$= P\left(Y_n \geq \theta - \frac{z}{n}\right)$$

$$= 1 - G_n(\theta - z/n)$$

$$= \begin{cases} 1 - \left(\frac{\theta - z/n}{\theta}\right)^n & 0 < \theta - z/n < \theta \\ 1 & \theta - z/n < 0 \\ 0 & \theta - z/n > \theta \end{cases}$$

$$H_n(z) = \begin{cases} 1 - \left(1 - \frac{z}{n\theta}\right)^n & 0 < z < n\theta \\ 1 & z > n\theta \\ 0 & z < 0 \end{cases}$$

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$$\lim_{n \to \infty} H_n(z) = \begin{cases} 1 - e^{-z/\theta} & z > 0 \\ 0 & z < 0 \end{cases}$$

This is $\text{Exp}(\beta = \theta)$

$\therefore \; Z_n \xrightarrow{D} Z,$

where $Z \sim \text{Exp}(\theta)$