

## Ordering conditional lifetimes of coherent systems

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### Abstract

Consider a system of  $n$  components that has the property that there exists a number  $r$  ( $r < n$ ), such that if it is known that at most  $r$  components have failed, the system is still functioning with probability 1. Suppose that such a system is equipped with a warning light that comes up at the time of the failure of the  $r$ th component. The system is still working then, and we are interested in its residual life. In this paper we obtain some results which stochastically compare the residual lives of such systems with the same type, or with different types, of components. Some applications are given. In particular, we derive upper and lower bounds on the expected residual lives of such systems given that the warning light has not come up yet, and given that the component hazard rate functions are bounded from below or from above by a known constant.

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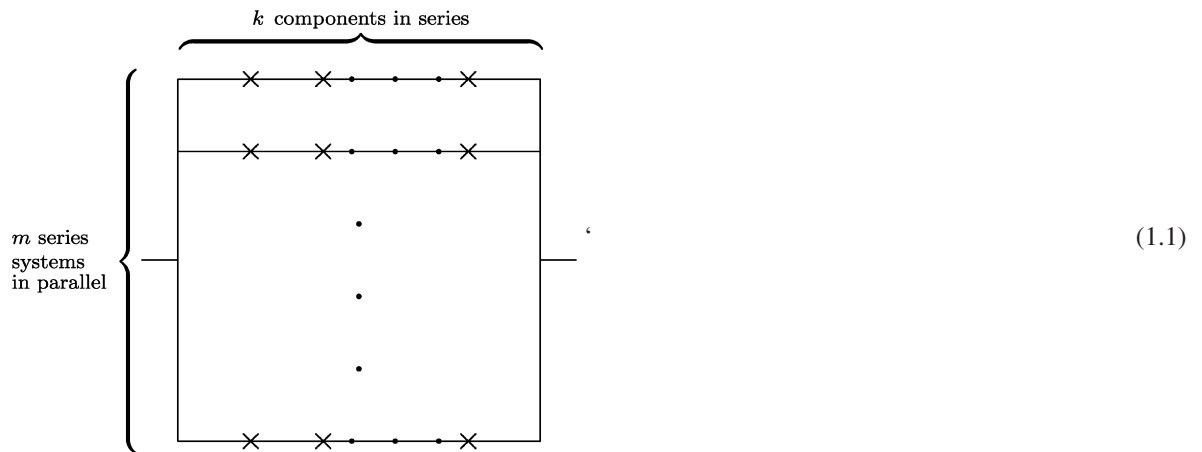
### 1. Introduction

Many coherent systems (see the definition and a thorough study of coherent systems in Barlow and Proschan, 1975) of size  $n$  have the property that there exists a certain number  $r$  ( $r < n$ ), such that if it is known that at most  $r$  components have failed, the system is still functioning with probability 1. For example, consider a  $k$ -out-of- $n$  system; if it is known that at most  $r$  ( $r < n - k + 1$ ) components have failed, then the system is still functioning with probability 1.

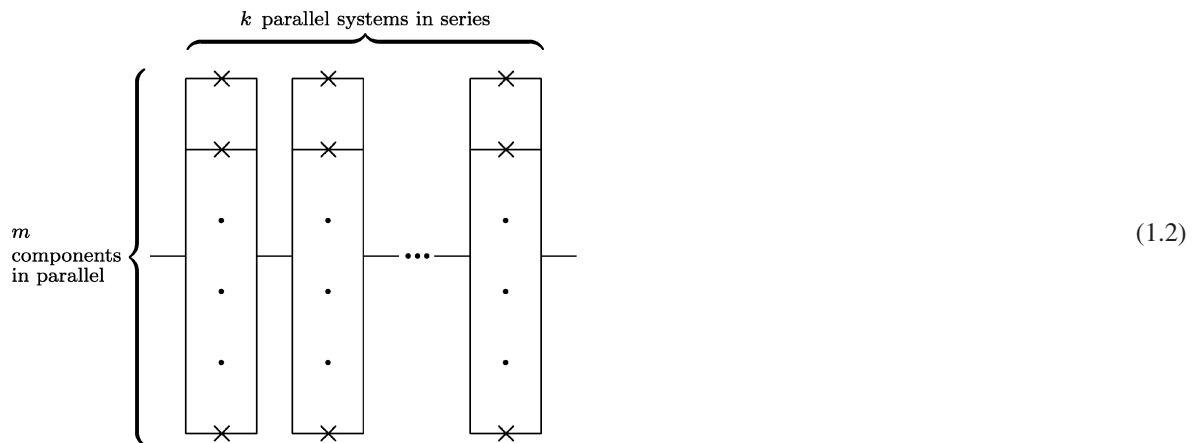
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A parallel-series system of the following form



also has this property; here if it is known that at most  $r$  ( $r < m$ ) components have failed, then the system is still functioning with probability 1. Similarly, a series-parallel system of the following form



has this property; again, here if it is known that at most  $r$  ( $r < m$ ) components have failed, then the system is still functioning with probability 1.

Suppose that a system of size  $n$ , as described above (that is, that works for sure if only  $r$  components have failed), is equipped with a warning light that comes up at the time of the failure of the  $r$ th component. The system is still working then, but the operator may now consider some maintenance procedure, or he may consider a replacement of the whole system. We thus see that it may be useful, for the operator's decision making, to know some properties of the residual life of the system before or after the light comes up, or to be able to stochastically compare then the residual lives of such systems with different types of components.

In this paper we obtain some results which stochastically compare the residual lives of such systems with the same type, or with different types, of components. Some applications are given in Section 3. In particular, we derive there upper and lower bounds on the expected residual lives of such systems given that the warning light has not come up yet, and given that the component hazard rate functions are bounded from below or from above by a known constant.

The following notation will be used throughout the paper. For any random variable  $X$ , and an event  $A$  with positive probability, we denote by  $[X|A]$  any random variable whose distribution function is the same as the conditional distribution function of  $X$  given  $A$ . For any collection of random variables  $X_1, X_2, \dots$ , we denote by  $X_{i:n}$  the  $i$ th order statistic among  $X_1, X_2, \dots, X_n$ .

The main building block that we use in this paper, in order to derive the main results, is the stochastic comparison of  $[X_{s:n} - y|X_{r:n} > y]$  and  $[Y_{s:n} - y|Y_{r:n} > y]$  where the  $X_{i:n}$ 's and the  $Y_{i:n}$ 's are order statistics from two random samples, and  $r \leq s \leq n$ . [Bairamov et al. \(2002\)](#) characterized some distribution functions by means of the expected value of the

above random quantities when  $r = 1$ , and Asadi and Bayramoglu (2005) studied some monotonicity properties of such expected values. Partial comparisons of the above random quantities can be found in the literature only when  $r = 1$  (see Asadi, 2004). However, when  $r > 1$ , the derivation of the stochastic comparison is more complex, when possible. In this paper we introduce methods that yield such a comparison.

It is worthwhile to mention that Li and Zuo (2002) and Li and Chen (2004) considered  $[X_{n-k+1:n} - X_{n-k:n} | X_{n-k:n} = t]$  and studied their aging properties. Li and Zuo (2004) compared  $[X - Y_1 | X > Y_1]$  with  $[X - Y_2 | X > Y_2]$  according to the ordinary stochastic order, where  $X, Y_1$ , and  $Y_2$  are mutually independent.

In this paper, ‘increasing’ stands for ‘non-decreasing’ and ‘decreasing’ stands for ‘non-increasing.’

## 2. Comparison of residual lives

In this section we first derive a basic result that stochastically compares residual lives of conditional order statistics.

For any two random variables  $X$  and  $Y$ , with respective distribution functions  $F$  and  $G$ , survival functions  $\bar{F} = 1 - F$  and  $\bar{G} = 1 - G$ , and right endpoints of support  $u_X \leq \infty$  and  $u_Y \leq \infty$ , we say that  $X$  is smaller than  $Y$  in the hazard rate order, denoted as  $X \leq_{hr} Y$ , if

$$\frac{\bar{G}(x)}{\bar{F}(x)} \text{ is increasing in } x \in (-\infty, \max\{u_X, u_Y\}),$$

where  $a/0$  is taken to be  $\infty$  wherever  $a > 0$ . For  $X$  and  $Y$  as above, we say that  $X$  is smaller than  $Y$  in the ordinary (usual) stochastic order, denoted as  $X \leq_{st} Y$ , if

$$\bar{F}(x) \leq \bar{G}(x) \text{ for all } x \in \mathbb{R}.$$

See, for example, Shaked and Shanthikumar (1994) or Müller and Stoyan (2002), for a detailed study of these stochastic orders. We recall that

$$X \leq_{hr} Y \implies X \leq_{st} Y. \tag{2.1}$$

**Theorem 2.1.** *Let  $X_1, X_2, \dots, X_n$  be a collection of independent and identically distributed random variables with a common distribution function  $F$ . Let  $Y_1, Y_2, \dots, Y_n$  be another collection of independent and identically distributed random variables with a common distribution function  $G$ . If  $X_1 \leq_{hr} Y_1$  then, for  $r \leq s \leq n$ , we have*

$$[X_{s:n} - y | X_{r:n} > y] \leq_{st} [Y_{s:n} - y | Y_{r:n} > y], \quad y \in \mathbb{R}. \tag{2.2}$$

In order to prove Theorem 2.1 we will need the following lemma. First recall that a discrete random variable  $X$ , with the discrete density function  $f$ , is said to be smaller in the likelihood ratio order than a discrete random variable  $Y$ , with the discrete density function  $g$ , denoted as  $X \leq_{lr} Y$ , if

$$\frac{g(x)}{f(x)} \text{ is increasing in } x \text{ on the union of the supports of } X \text{ and } Y,$$

where, again,  $a/0$  is taken to be  $\infty$  wherever  $a > 0$ . Also, recall that

$$X \leq_{lr} Y \implies X \leq_{st} Y. \tag{2.3}$$

**Lemma 2.2.** *Let  $Z_1$  and  $Z_2$  be two binomial random variables with parameters  $(n, p_1)$  and  $(n, p_2)$ , respectively. If  $p_1 \leq p_2$  then  $[Z_1 | Z_1 \geq r] \leq_{lr} [Z_2 | Z_2 \geq r]$ ,  $r = 0, 1, \dots, n$ .*

The proof of Lemma 2.2 is elementary, and is omitted.

**Proof of Theorem 2.1.** For fixed  $r \leq s \leq n$  and  $y$ , the distribution function of  $[X_{s:n} - y | X_{r:n} > y]$  can be written, for  $x \geq 0$ , as

$$\begin{aligned} P\{X_{s:n} \leq x + y | X_{r:n} > y\} &= \frac{P\{y < X_{r:n} \leq X_{s:n} \leq x + y\}}{P\{X_{r:n} > y\}} \\ &= \frac{\sum_{j=n-r+1}^n \sum_{l=s-n+j}^j n! / ((n-j)! (j-l)!) F^{n-j}(y) (F(x+y) - F(y))^l \bar{F}^{j-l}(x+y)}{\sum_{k=n-r+1}^n \binom{n}{k} F^{n-k}(y) \bar{F}^k(y)}. \end{aligned}$$

That is,

$$\begin{aligned} P\{X_{s:n} \leq x + y | X_{r:n} > y\} &= \sum_{j=n-r+1}^n \frac{\binom{n}{j} F^{n-j}(y) \sum_{l=s-n+j}^j \binom{j}{l} (F(x+y) - F(y))^l \bar{F}^{j-l}(x+y)}{\sum_{k=n-r+1}^n \binom{n}{k} F^{n-k}(y) \bar{F}^k(y)} \\ &= \sum_{j=n-r+1}^n \frac{\binom{n}{j} F^{n-j}(y) \bar{F}^j(y) \sum_{l=s-n+j}^j \binom{j}{l} (1 - \bar{F}(x+y) / \bar{F}(y))^l (\bar{F}(x+y) / \bar{F}(y))^{j-l}}{\sum_{k=n-r+1}^n \binom{n}{k} F^{n-k}(y) \bar{F}^k(y)}. \end{aligned}$$

Let  $Z_1$  denote a binomial random variable with parameters  $(n, \bar{F}(y))$ , and let  $p_j = P\{Z_1 = j | Z_1 \geq n - r + 1\}$ . Then

$$\begin{aligned} P\{X_{s:n} \leq x + y | X_{r:n} > y\} &= \sum_{j=n-r+1}^n p_j \sum_{l=s-n+j}^j \binom{j}{l} \left(1 - \frac{\bar{F}(x+y)}{\bar{F}(y)}\right)^l \left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)^{j-l} \\ &= \sum_{j=n-r+1}^n p_j F_{X_{s-n+j:j}^y}(x), \quad x \geq 0, \end{aligned} \tag{2.4}$$

where  $X_{s-n+j:j}^y$  denotes the  $(s - n + j)$ th order statistic among the independent and identically distributed random variables  $X_1^y, X_2^y, \dots, X_j^y$  that are all distributed as  $[X_1 - y | X_1 > y]$ , and  $F_{X_{s-n+j:j}^y}$  denotes the distribution function of  $X_{s-n+j:j}^y$ ; that is,  $X_{s-n+j:j}^y$  is the  $(s - n + j)$ th order statistic in a sample from  $F$  truncated on the left at  $y$ .

Similarly, the distribution function of  $[Y_{s:n} - y | Y_{r:n} > y]$  can be written as

$$P\{Y_{s:n} \leq x + y | Y_{r:n} > y\} = \sum_{j=n-r+1}^n q_j G_{Y_{s-n+j:j}^y}(x), \quad x \geq 0;$$

here  $q_j = P\{Z_2 = j | Z_2 \geq n - r + 1\}$  where  $Z_2$  is a binomial random variable with parameters  $(n, \bar{G}(y))$ ,  $Y_{s-n+j:j}^y$  denotes the  $(s - n + j)$ th order statistic among the independent and identically distributed random variables  $Y_1^y, Y_2^y, \dots, Y_j^y$  that are all distributed as  $[Y_1 - y | Y_1 > y]$ , and  $G_{Y_{s-n+j:j}^y}$  denotes the distribution function of  $Y_{s-n+j:j}^y$ .

We now make the following observations. The assumption  $X_1 \leq_{hr} Y_1$  implies, by (2.1), that  $X_1 \leq_{st} Y_1$ , that is,  $\bar{F}(y) \leq \bar{G}(y)$ . Therefore, by Lemma 2.2 and (2.3),

$$[Z_1 | Z_1 \geq n - r + 1] \leq_{st} [Z_2 | Z_2 \geq n - r + 1]. \tag{2.5}$$

The assumption  $X_1 \leq_{hr} Y_1$  also implies that  $[X_1 - y | X_1 > y] \leq_{st} [Y_1 - y | Y_1 > y]$ . Hence

$$X_{s-n+j:j}^y \leq_{st} Y_{s-n+j:j}^y, \quad j = n - r + 1, n - r + 2, \dots, n,$$

and therefore

$$F_{X_{s-n+j:j}^y}(x) \geq G_{Y_{s-n+j:j}^y}(x), \quad x \geq 0. \tag{2.6}$$

Finally, from results of Mi and Shaked (2002) or of Korwar (2003) it follows that

$$Y_{s-n+j:j}^y \uparrow_{st} j, \quad j = n - r + 1, n - r + 2, \dots, n,$$

and therefore

$$G_{Y_{s-n+j:j}^y}(x) \text{ is decreasing in } j = n - r + 1, n - r + 2, \dots, n, \quad x \geq 0. \tag{2.7}$$

Combining these observations we have, for  $x \geq 0$ , that

$$\begin{aligned} P\{X_{s:n} \leq x + y | X_{r:n} > y\} &= \sum_{j=n-r+1}^n p_j F_{X_{s-n+j:j}^y}(x) \\ &\geq \sum_{j=n-r+1}^n p_j G_{Y_{s-n+j:j}^y}(x) \quad (\text{by (2.6)}) \\ &\geq \sum_{j=n-r+1}^n q_j G_{Y_{s-n+j:j}^y}(x) \quad (\text{by (2.5) and (2.7)}) \\ &= P\{Y_{s:n} \leq x + y | Y_{r:n} > y\}, \end{aligned}$$

where the second inequality is obtained from the characterization of the order  $\leq_{st}$  given in expression (1.A.7) of Shaked and Shanthikumar (1994). This completes the proof.  $\square$

Theorem 2.1 enables us to compare stochastically the residual lifetimes of two  $k$ -out-of- $n$  systems that each has identical components, but where the components of one system are different than the components of the other system. Explicitly, we can stochastically compare the residual lives of such systems if it is known that the lifetime of a component in one system is smaller than the counterpart in the second system in the hazard rate ordering, and that at time  $t$ , say, at most  $m$  ( $m < n - k + 1$ ) of the components have already failed; just take  $y = t$ ,  $r = m + 1$ , and  $s = n - k + 1$  in (2.2).

However, Theorem 2.1 does not let us compare stochastically systems such as those described in (1.1) and (1.2). In order to compare such systems we need a result that is more general than Theorem 2.1.

Recall that an  $n$ -component system is called *coherent* if the system is monotone (that is, an improvement of a component cannot lead to a deterioration in system performance) and if every component is relevant. For a coherent system with component lifetimes  $z_1, z_2, \dots, z_n$ , let  $\tau(z_1, z_2, \dots, z_n)$  denote the lifetime of the system—such a function  $\tau$  is called a *coherent life function*; see Esary and Marshall (1970) for a basic study of coherent life functions. Samaniego (1985) introduced, and Kochar et al. (1999) further studied, a useful concept that can be used to express the distribution function of the lifetime of a coherent system with independent and identically distributed continuous component lifetimes  $Z_1, Z_2, \dots, Z_n$ . They observed that the distribution function, of the lifetime  $\tau(Z_1, Z_2, \dots, Z_n)$  of a coherent system, can be expressed as a mixture of the distribution functions of the order statistics  $Z_{1:n}, Z_{2:n}, \dots, Z_{n:n}$ . Explicitly, they defined the ‘signature’ of  $\tau$  as the probability vector  $\mathbf{p} = (p_1, p_2, \dots, p_n)$  with elements

$$p_i = P\{\tau(Z_1, Z_2, \dots, Z_n) = Z_{i:n}\}, \quad i = 1, 2, \dots, n.$$

Note that  $\mathbf{p}$  is a genuine probability vector because  $\tau(Z_1, Z_2, \dots, Z_n) \in \{Z_{1:n}, Z_{2:n}, \dots, Z_{n:n}\}$  with probability 1. From the continuity of the component lifetimes it follows that  $\mathbf{p}$  does not depend on the common distribution function of  $Z_1, Z_2, \dots, Z_n$ . Thus, the distribution function of  $\tau(Z_1, Z_2, \dots, Z_n)$  is a mixture of the distribution functions of  $Z_{1:n}, Z_{2:n}, \dots, Z_{n:n}$  with the respective weights  $p_1, p_2, \dots, p_n$ . The weights  $p_1, p_2, \dots, p_n$  can be computed by

$$p_i = \frac{n_i}{n!}, \quad i = 1, 2, \dots, n,$$

where  $n_i$  is the number of ways that distinct  $z_1, z_2, \dots, z_n$  can be ordered such that  $\tau(z_1, z_2, \dots, z_n) = z_{i:n}$ ,  $i = 1, 2, \dots, n$ ; here  $z_{i:n}$  denotes the  $i$ th smallest value among  $z_1, z_2, \dots, z_n$ .

In Navarro et al. (2005) it is observed that the distribution function of  $\tau(Z_1, Z_2, \dots, Z_n)$  is a mixture of the distribution functions of  $Z_{1:n}, Z_{2:n}, \dots, Z_{n:n}$  with the respective weights  $p_1, p_2, \dots, p_n$ , even if  $Z_1, Z_2, \dots, Z_n$ , with some absolutely continuous distribution function, are exchangeable (rather than merely independent and identically distributed).

Note that if a system has the property that, with probability 1, it is alive as long as at least  $n - s + 1$  components are alive, then its signature  $\mathbf{p}$  must be of the form  $(0, 0, \dots, 0, p_s, p_{s+1}, \dots, p_n)$ . In the context of this paper, we can consider such systems with a warning light that comes up at the time of the failure of the  $r$ th component,  $r = 1, 2, \dots, s - 1$ . There are a lot of systems with such signatures where  $s > 1$ . See, for example, Shaked and Suarez-Llorens (2003) for a list of such systems when  $n \leq 4$ . See (1.1) and (1.2), and Example 3.5 below, for other systems with such signatures.

For any two probability vectors  $\mathbf{p} = (p_1, p_2, \dots, p_n)$  and  $\mathbf{q} = (q_1, q_2, \dots, q_n)$  we denote  $\mathbf{p} \leq_{st} \mathbf{q}$  if  $\sum_{i=j}^n p_i \leq \sum_{i=j}^n q_i$ ,  $j = 1, 2, \dots, n$ .

**Theorem 2.3.** *Let  $X_1, X_2, \dots, X_n$  be the independent and identically distributed continuous lifetimes of the components of a coherent system with life function  $\tau_1$ , and let  $Y_1, Y_2, \dots, Y_n$  be the independent and identically distributed continuous lifetimes of the components of another coherent system with life function  $\tau_2$ . Denote  $T_X = \tau_1(X_1, X_2, \dots, X_n)$  and  $T_Y = \tau_2(Y_1, Y_2, \dots, Y_n)$ . For some  $1 \leq s \leq n$ , suppose that the signature of  $\tau_1$  is of the form  $\mathbf{p} = (0, 0, \dots, 0, p_s, p_{s+1}, \dots, p_n)$ , and that the signature of  $\tau_2$  is of the form  $\mathbf{q} = (0, 0, \dots, 0, q_s, q_{s+1}, \dots, q_n)$ . If  $\mathbf{p} \leq_{st} \mathbf{q}$  and if  $X_1 \leq_{hr} Y_1$  then, for  $r \leq s$ , we have*

$$[T_X - y | X_{r:n} > y] \leq_{st} [T_Y - y | Y_{r:n} > y], \quad y \geq 0.$$

**Proof.** For  $s \leq i \leq n$ , let  $A_i$  be the set of all the permutations  $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$  of  $(1, 2, \dots, n)$  such that if  $X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}$  then  $T_X = X_{\pi_i}$ . That is,  $A_i$  is the set of all the permutations  $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$  such that if  $X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}$  then  $T_X = X_{i:n}$ . Thus, for  $x \geq 0$  and  $y \geq 0$ , we have

$$\begin{aligned} P\{T_X - y > x | X_{r:n} > y\} &= \sum_{i=s}^n P\{T_X - y > x, T_X = X_{i:n} | X_{r:n} > y\} \\ &= \sum_{i=s}^n \sum_{\boldsymbol{\pi} \in A_i} P\{T_X - y > x, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n} | X_{r:n} > y\} \\ &= \sum_{i=s}^n \sum_{\boldsymbol{\pi} \in A_i} P\{X_{\pi_i} - y > x, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n} | X_{r:n} > y\} \\ &= \sum_{i=s}^n \sum_{\boldsymbol{\pi} \in A_i} P\{X_{i:n} - y > x, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n} | X_{r:n} > y\}. \end{aligned}$$

Now note, by the exchangeability of  $X_1, X_2, \dots, X_n$ , that for any permutation  $\boldsymbol{\pi}$  we have

$$\begin{aligned} &P\{X_{i:n} - y > x, X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n} | X_{r:n} > y\} \\ &= P\{X_{i:n} - y > x | X_{r:n} > y\} P\{X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n} | X_{r:n} > y\} \\ &= P\{X_{i:n} - y > x | X_{r:n} > y\} P\{X_{\pi_1} < X_{\pi_2} < \dots < X_{\pi_n}\} \\ &= P\{X_{i:n} - y > x | X_{r:n} > y\} \cdot \frac{1}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} P\{T_X - y > x | X_{r:n} > y\} &= \sum_{i=s}^n \left( \sum_{\boldsymbol{\pi} \in A_i} \frac{1}{n!} \right) P\{X_{i:n} - y > x | X_{r:n} > y\} \\ &= \sum_{i=s}^n p_i P\{X_{i:n} - y > x | X_{r:n} > y\}. \end{aligned} \tag{2.8}$$

Similarly,

$$P\{T_Y - y > x | Y_{r:n} > y\} = \sum_{i=s}^n q_i P\{Y_{i:n} - y > x | Y_{r:n} > y\}.$$

Hence

$$\begin{aligned} P\{T_X - y > x | X_{r:n} > y\} &= \sum_{i=s}^n p_i P\{X_{i:n} - y > x | X_{r:n} > y\} \\ &\leq \sum_{i=s}^n p_i P\{Y_{i:n} - y > x | Y_{r:n} > y\} \quad (\text{by (2.2)}) \\ &\leq \sum_{i=s}^n q_i P\{Y_{i:n} - y > x | Y_{r:n} > y\} \quad (\text{by } \mathbf{p} \leq_{st} \mathbf{q}), \end{aligned}$$

where the last inequality follows from the fact that  $P\{Y_{i:n} - y > x | Y_{r:n} > y\}$  is increasing in  $i \in \{s, s + 1, \dots, n\}$  (see, for example, (2.4)), and from the characterization of the order  $\leq_{st}$  given in expression (1.A.7) of Shaked and Shanthikumar (1994). This completes the proof.  $\square$

As a corollary of Theorem 2.3 we have the following result which yields a stochastic comparison of the residual lives of the same system when it has all components of one of two types.

**Corollary 2.4.** *Let  $\tau$  be an  $n$ -component coherent life function with signature of the form  $\mathbf{p} = (0, 0, \dots, 0, p_s, p_{s+1}, \dots, p_n)$ . Let  $X_1, X_2, \dots, X_n$  be the independent and identically distributed continuous lifetimes of one type of components for this coherent system, and let  $Y_1, Y_2, \dots, Y_n$  be the independent and identically distributed continuous lifetimes of another type of components for this coherent system. Denote  $T_X = \tau(X_1, X_2, \dots, X_n)$  and  $T_Y = \tau(Y_1, Y_2, \dots, Y_n)$ . If  $X_1 \leq_{hr} Y_1$  then, for  $r \leq s$ , we have*

$$[T_X - y | X_{r:n} > y] \leq_{st} [T_Y - y | Y_{r:n} > y], \quad y \geq 0.$$

We end this section with an analog of Theorem 2.3 which involves inactivity times (rather than residual lives). We recall that for any two random variables  $X$  and  $Y$ , with respective distribution functions  $F$  and  $G$ , and left endpoints of support  $l_X \geq -\infty$  and  $l_Y \geq -\infty$ , we say that  $X$  is smaller than  $Y$  in the reversed hazard rate order, denoted as  $X \leq_{rh} Y$ , if

$$\frac{G(x)}{F(x)} \text{ is increasing in } x \in (-\infty, \min\{l_X, l_Y\}),$$

where  $a/0$  is taken to be  $\infty$  whenever  $a > 0$ .

**Theorem 2.5.** *Let  $X_1, X_2, \dots, X_n$  be the independent and identically distributed continuous lifetimes of the components of a coherent system with life function  $\tau_1$ , and let  $Y_1, Y_2, \dots, Y_n$  be the independent and identically distributed continuous lifetimes of the components of another coherent system with life function  $\tau_2$ . Denote  $T_X = \tau_1(X_1, X_2, \dots, X_n)$  and  $T_Y = \tau_2(Y_1, Y_2, \dots, Y_n)$ . For some  $1 \leq s \leq n$ , suppose that the signature of  $\tau_1$  is of the form  $\mathbf{p} = (p_1, p_2, \dots, p_s, 0, 0, \dots, 0)$ , and that the signature of  $\tau_2$  is of the form  $\mathbf{q} = (q_1, q_2, \dots, q_s, 0, 0, \dots, 0)$ . If  $\mathbf{p} \leq_{st} \mathbf{q}$  and if  $X_1 \leq_{rh} Y_1$  then, for  $r \geq s$ , we have*

$$[y - T_X | X_{r:n} \leq y] \geq_{st} [y - T_Y | Y_{r:n} \leq y], \quad y \geq 0.$$

**Proof.** If  $X_1 \leq_{rh} Y_1$  then  $-X_1 \geq_{hr} -Y_1$ ; see, for example, Nanda and Shaked (2001). Let  $(-X)_{s:n}$  denote the  $s$ th order statistic among  $-X_1, -X_2, \dots, -X_n$ , and let  $(-X)_{r:n}, (-Y)_{s:n}$ , and  $(-Y)_{r:n}$  be similarly defined. From Theorem 2.1 it follows that

$$[(-X)_{s:n} - y | (-X)_{r:n} > y] \geq_{st} [(-Y)_{s:n} - y | (-Y)_{r:n} > y], \quad r \leq s, \quad y \in \mathbb{R};$$

that is,

$$[-X_{n-s+1:n} | -X_{n-r+1:n} > y] \geq_{st} [-Y_{n-s+1:n} | -Y_{n-r+1:n} > y], \quad r \leq s, \quad y \in \mathbb{R};$$

that is,

$$[X_{n-s+1:n} | X_{n-r+1:n} \leq z] \leq_{st} [Y_{n-s+1:n} | Y_{n-r+1:n} \leq z], \quad r \leq s, \quad z \in \mathbb{R};$$

that is,

$$[X_{k:n} | X_{l:n} \leq z] \leq_{st} [Y_{k:n} | Y_{l:n} \leq z], \quad l \geq k, \quad z \in \mathbb{R};$$

that is,

$$[z - X_{k:n} | X_{l:n} \leq z] \geq_{st} [z - Y_{k:n} | Y_{l:n} \leq z], \quad l \geq k, \quad z \in \mathbb{R}.$$

Thus we have an analog of Theorem 2.1 under the assumption  $X_1 \leq_{rh} Y_1$ .

The rest of the proof of Theorem 2.5 is similar to the way that Theorem 2.3 is obtained from Theorem 2.1; we omit the details.  $\square$

### 3. Applications

First we state a proposition that will be used in the sequel.

**Proposition 3.1.** *Let  $X_1, X_2, \dots, X_n$  be a collection of independent and identically distributed exponential random variables with hazard rate  $\lambda$ . For  $y \geq 0$ , let  $Z$  denote a binomial random variable with parameters  $(n, e^{-\lambda y})$ . Then, for  $r \leq s \leq n$ , we have*

$$E[X_{s:n} - y | X_{r:n} > y] = \frac{1}{\lambda} \sum_{j=n-r+1}^n p_j \sum_{k=n-s+1}^j \frac{1}{k}, \tag{3.1}$$

where  $p_j = P\{Z = j | Z \geq n - r + 1\}$ ,  $j = n - r + 1, n - r + 2, \dots, n$ ; that is,

$$p_j = \frac{\binom{n}{j} e^{-j\lambda y} (1 - e^{-\lambda y})^{n-j}}{\sum_{l=n-r+1}^n \binom{n}{l} e^{-l\lambda y} (1 - e^{-\lambda y})^{n-l}}, \quad j = n - r + 1, n - r + 2, \dots, n. \tag{3.2}$$

**Proof.** As in the proof of Theorem 2.1, let  $X_{s-n+j:j}^y$  denote the  $(s - n + j)$ th order statistic among the independent and identically distributed random variables  $X_1^y, X_2^y, \dots, X_j^y$  that are all distributed as  $[X_1 - y | X_1 > y]$ . These are independent and identically distributed exponential random variables, and therefore

$$EX_{s-n+j:j}^y = \frac{1}{\lambda j} + \frac{1}{\lambda(j-1)} + \dots + \frac{1}{\lambda(n-s+1)}, \quad j = n - r + 1, n - r + 2, \dots, n;$$

see Barlow and Proschan (1975, p. 60). Now, from (2.4) we get that

$$E[X_{s:n} - y | X_{r:n} > y] = \sum_{j=n-r+1}^n p_j EX_{s-n+j:j}^y.$$

Plugging the previous expression for  $EX_{s-n+j:j}^y$  in the above equality, gives (3.1).  $\square$

As a consequence of Proposition 3.1 we obtain the following result.

**Proposition 3.2.** *Let  $\tau$  be a coherent life function with signature of the form  $\mathbf{q} = (0, 0, \dots, 0, q_s, q_{s+1}, \dots, q_n)$ . Let the lifetimes  $X_1, X_2, \dots, X_n$  of the components of the system be independent and identically distributed exponential*

random variables with hazard rate  $\lambda$ . Denote  $T_X = \tau(X_1, X_2, \dots, X_n)$ . Then, for  $y \geq 0$  and  $r \leq s \leq n$ , we have

$$E[T_X - y | X_{r:n} > y] = \frac{1}{\lambda} \sum_{i=s}^n \sum_{j=n-r+1}^n q_i p_j \sum_{k=n-i+1}^j \frac{1}{k},$$

where  $p_j, j = n - r + 1, n - r + 2, \dots, n$ , are given in (3.2).

**Proof.** From (2.8) we see that

$$E[T_X - y | X_{r:n} > y] = \sum_{i=s}^n q_i E[X_{i:n} - y | X_{r:n} > y]$$

(note that the  $q_i$ 's here play the role of the  $p_i$ 's in (2.8)). The stated result now follows from (3.1).  $\square$

Corollary 2.4 and Proposition 3.2 enable us to obtain computable bounds on the expected conditional residual life of a coherent system that has identical components with independent lifetimes. Explicitly, suppose that the common lifetime distribution of the components is not known in detail, but suppose that it is known that its hazard rate function is bounded from above by some constant  $\lambda$ , say. Furthermore, suppose that the system has a signature of the form  $q = (0, 0, \dots, 0, q_s, q_{s+1}, \dots, q_n)$ . Let  $Y_1, Y_2, \dots, Y_n$  denote the lifetimes of the components, and let  $X_1, X_2, \dots, X_n$  be independent exponential random variables with hazard rate  $\lambda$ . The fact that the hazard rate function of the  $Y_i$ 's is bounded from above by  $\lambda$  means that  $X_1 \leq_{hr} Y_1$ . Let  $T_Y$  denote the actual lifetime of the system, and let  $T_X$  denote the lifetime of the system if the component lifetimes were  $X_1, X_2, \dots, X_n$ . By Corollary 2.4, for  $r \leq s$ , we have that

$$E[T_Y - y | Y_{r:n} > y] \geq E[T_X - y | X_{r:n} > y].$$

Thus, from Proposition 3.2 we obtain a lower bound on  $E[T_Y - y | Y_{r:n} > y]$  as follows:

$$E[T_Y - y | Y_{r:n} > y] \geq \frac{1}{\lambda} \sum_{i=s}^n \sum_{j=n-r+1}^n q_i p_j \sum_{k=n-i+1}^j \frac{1}{k}, \tag{3.3}$$

where  $p_j, j = n - r + 1, n - r + 2, \dots, n$ , are given in (3.2).

Similarly, if it is known that the hazard rate function of the component lifetimes is bounded from below by some constant  $\lambda$ , then, from Corollary 2.4 and Proposition 3.2, we obtain an upper bound on  $E[T_Y - y | Y_{r:n} > y]$  as follows:

$$E[T_Y - y | Y_{r:n} > y] \leq \frac{1}{\lambda} \sum_{i=s}^n \sum_{j=n-r+1}^n q_i p_j \sum_{k=n-i+1}^j \frac{1}{k},$$

where  $p_j, j = n - r + 1, n - r + 2, \dots, n$ , are given in (3.2).

Note that the bounds above are attained when the components have exponential distributions.

**Example 3.3.** Consider a 2-out-of-4 system with a warning light that comes up at the time of the second failure of a component (note the system fails at the time of the third failure). Suppose that it is known that the failure rate function of each of the (identical) components is bounded from above by some (fixed)  $\lambda > 0$ . Suppose also that at some (fixed) time  $y > 0$  the warning light has not come up yet. We will now compute the lower bound (that is, the right-hand side of (3.3)) on the expected conditional residual life of the system given the above information.

Note that the warning light being off at time  $y$  means that  $Y_{2:4} > y$  where  $Y_1, Y_2, Y_3, Y_4$  are the component lifetimes. The signature of the system here is  $(0, 0, 1, 0)$ ; that is,  $q_3 = 1$  and all the other  $q_i$ 's are 0. Thus, the right-hand side of (3.3) reduces to the right-hand side of (3.1) with  $n = 4, s = 3$ , and  $r = 2$ . Explicitly, the lower bound is

$$\frac{1}{\lambda} \left[ \frac{5}{6} \cdot p_3 + \frac{13}{12} \cdot p_4 \right],$$

where

$$p_3 = \frac{\binom{4}{3} e^{-3\lambda y} (1 - e^{-\lambda y})^{4-3}}{\sum_{l=3}^4 \binom{4}{l} e^{-l\lambda y} (1 - e^{-\lambda y})^{4-l}} = \frac{4(1 - e^{-\lambda y})}{4(1 - e^{-\lambda y}) + e^{-\lambda y}} \tag{3.4}$$

and

$$p_4 = \frac{\binom{4}{4} e^{-4\lambda y} (1 - e^{-\lambda y})^{4-4}}{\sum_{l=3}^4 \binom{4}{l} e^{-l\lambda y} (1 - e^{-\lambda y})^{4-l}} = \frac{e^{-\lambda y}}{4(1 - e^{-\lambda y}) + e^{-\lambda y}}. \tag{3.5}$$

For example, suppose that  $\lambda = 1$  and that the warning light has not come up yet by time  $y = 1$ . Denoting the lifetime of the system by  $T_Y$  we have

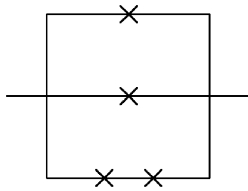
$$E[T_Y - 1 | Y_{2:4} > 1] \geq \frac{5}{6} \cdot \frac{4(1 - e^{-1})}{4(1 - e^{-1}) + e^{-1}} + \frac{13}{12} \cdot \frac{e^{-1}}{4(1 - e^{-1}) + e^{-1}} \approx 0.865087.$$

If the warning light has not come up yet by time  $y = 2$  then

$$E[T_Y - 2 | Y_{2:4} > 2] \geq \frac{5}{6} \cdot \frac{4(1 - e^{-2})}{4(1 - e^{-2}) + e^{-2}} + \frac{13}{12} \cdot \frac{e^{-2}}{4(1 - e^{-2}) + e^{-2}} \approx 0.842747.$$

If it is known that the failure rate function of each of the components is bounded from below (rather than from above) by  $\lambda$ , then the above inequalities are reversed; that is, the resulting bounds are then upper bounds.

**Example 3.4.** Consider the following coherent system



with a warning light that comes up at the time of the second failure of a component. Suppose that it is known that the failure rate function of each of the (identical) components is bounded from above by some  $\lambda > 0$ . Suppose also that at some time  $y > 0$  the warning light has not come up yet; note that means that  $Y_{2:4} > y$  where  $Y_1, Y_2, Y_3, Y_4$  are the component lifetimes. The signature of the system here is  $(0, 0, 0.5, 0.5)$ ; that is,  $q_3 = q_4 = 0.5$ . The right-hand side of (3.3) with  $n = 4, s = 3$ , and  $r = 2$ , is

$$\frac{1}{\lambda} \left[ \frac{4}{3} \cdot p_3 + \frac{19}{12} \cdot p_4 \right],$$

where  $p_3$  and  $p_4$  are given in (3.4) and (3.5).

For example, suppose that  $\lambda = 1$  and that the warning light has not come up yet by time  $y = 1$ . Denoting the lifetime of the system by  $T_Y$  we have

$$E[T_Y - 1 | Y_{2:4} > 1] \geq \frac{4}{3} \cdot \frac{4(1 - e^{-1})}{4(1 - e^{-1}) + e^{-1}} + \frac{19}{12} \cdot \frac{e^{-1}}{4(1 - e^{-1}) + e^{-1}} \approx 1.36509.$$

Similarly, if the warning light has not come up yet by time  $y = 2$  then  $E[T_Y - 2 | Y_{2:4} > 2] \geq 1.34275$ .

**Example 3.5.** Consider the system in (1.1) with  $m = 3$  and  $k = 2$ . Suppose that a warning light comes up at the time of the second failure of a component. As in the previous examples, suppose that it is known that the failure rate function of each of the (identical) components is bounded from above by some  $\lambda > 0$ . Consider the situation in which, at some time  $y > 0$ , the warning light has not come up yet; note that means that  $Y_{2:6} > y$  where  $Y_1, Y_2, \dots, Y_6$  are the component

lifetimes. The signature of the system here is  $(0, 0, 0.4, 0.4, 0.2, 0)$ ; that is,  $q_3 = q_4 = 0.4$  and  $q_5 = 0.2$ . The right-hand side of (3.3) with  $n = 6$ ,  $s = 3$ , and  $r = 2$  is

$$\frac{1}{\lambda} \left[ \frac{3}{4} \cdot p_5 + \frac{11}{12} \cdot p_6 \right],$$

where

$$p_5 = \frac{\binom{6}{5} e^{-5\lambda y} (1 - e^{-\lambda y})^{6-5}}{\sum_{l=5}^6 \binom{6}{l} e^{-l\lambda y} (1 - e^{-\lambda y})^{6-l}} = \frac{6(1 - e^{-\lambda y})}{6(1 - e^{-\lambda y}) + e^{-\lambda y}}$$

and

$$p_6 = \frac{\binom{6}{6} e^{-6\lambda y} (1 - e^{-\lambda y})^{6-6}}{\sum_{l=5}^6 \binom{6}{l} e^{-l\lambda y} (1 - e^{-\lambda y})^{6-l}} = \frac{e^{-\lambda y}}{6(1 - e^{-\lambda y}) + e^{-\lambda y}}.$$

For example, suppose that  $\lambda = 1$  and that the warning light has not come up yet by time  $y = 1$ . Denoting the lifetime of the system by  $T_Y$  we have

$$E[T_Y - 1 | Y_{2:6} > 1] \geq \frac{3}{4} \cdot \frac{6(1 - e^{-1})}{6(1 - e^{-1}) + e^{-1}} + \frac{11}{12} \cdot \frac{e^{-1}}{6(1 - e^{-1}) + e^{-1}} \approx 0.764737.$$

Similarly, if the warning light has not come up yet by time  $y = 2$  then  $E[T_Y - 2 | Y_{2:6} > 2] \geq 0.754237$ .

The above examples show how our results yield simple-to-compute bounds on the expected conditional residual lives of quite complex coherent systems that are equipped with a warning light, if it is known that the component’s hazard rate function is bounded by a known constant.

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**References**

Asadi, M., 2004. On the mean residual life and the mean past life of the  $k$ -out-of- $n$  systems. In: Proceedings of the 7th Iranian Statistical Conference, Invited Papers Volume. Allameh Tabatabaie University, Tehran, pp. 55–77.

Asadi, M., Bayramoglu, I., 2005. A note on the mean residual life function of a parallel system. *Comm. Statist. Theory Methods* 34, 475–484.

Bairamov, I., Ahsanullah, M., Akhundov, I., 2002. A residual life function of a system having parallel or series structures. *J. Statist. Theory Appl.* 1, 119–132.

Barlow, R.E., Proschan, F., 1975. *Statistical Theory of Reliability and Life Testing: Reliability Models*. Holt, Rinehart and Winston, New York.

Esary, J.D., Marshall, A.W., 1970. Coherent life functions. *SIAM J. Appl. Math.* 18, 810–814.

Kochar, S.C., Mukerjee, H., Samaniego, F.J., 1999. The ‘signature’ of a coherent system and its application to comparisons among systems. *Naval Res. Logist.* 46, 507–523.

Korwar, R., 2003. On stochastic orders for the lifetime of a  $k$ -out-of- $n$  system. *Probab. Eng. Inform. Sci.* 17, 137–142.

Li, X., Chen, J., 2004. Aging properties of the residual life length of  $k$ -out-of- $n$  systems with independent but non-identical components. *Appl. Stochastic Models Business Industry* 20, 143–153.

Li, X., Zuo, M.J., 2002. On the behaviour of some new ageing properties based upon the residual life of  $k$ -out-of- $n$  systems. *J. Appl. Probab.* 39, 426–433.

Li, X., Zuo, M.J., 2004. Stochastic comparison of residual life and inactivity time at a random time. *Stochastic Models* 20, 229–235.

Mi, J., Shaked, M., 2002. Stochastic dominance of random variables implies the dominance of their order statistics. *J. Indian Statist. Assoc.* 40, 161–168.

Müller, A., Stoyan, D., 2002. *Comparison Methods for Stochastic Models and Risks*. Wiley, New York.

Nanda, A.K., Shaked, M., 2001. The hazard rate and the reversed hazard rate orders, with applications to order statistics. *Ann. Inst. Statist. Math.* 53, 853–864.

- Navarro, J., Ruiz, J.M., Sandoval, C.J., 2005. A note on comparisons among coherent systems with dependent components using signatures. *Statist. Probab. Lett.* 72, 179–185.
- Samaniego, F.J., 1985. On the closure of the IFR class under formation of coherent systems. *IEEE Trans. Reliability* R-34, 69–72.
- Shaked, M., Shanthikumar, J.G., 1994. *Stochastic Orders and their Applications*. Academic Press, Boston.
- Shaked, M., Suarez-Llorens, A., 2003. On the comparison of reliability experiments based on the convolution order. *J. Amer. Statist. Assoc.* 98, 693–702.