

MATH 413/513

Real Analysis III

HOMEWORK ASSIGNMENT 4

Due Wednesday, May 20

- (1) A norm $\|\cdot\|$ is called strictly convex if $\|x\| = 1$, $\|y\| = 1$, and $x \neq y$, imply $\|\frac{x+y}{2}\| < 1$. (Draw some pictures to convince yourself that in \mathbb{R}^2 , $\|\cdot\|_2$ is strictly convex while $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not.) Prove that in a Hilbert space the induced norm is always strictly convex.
- (2) ★ In a Hilbert space H , we call convergence in the norm **strong convergence**. That is, a sequence $\{x_n\}$ converges to a point x strongly if $\|x_n - x\| \rightarrow 0$. We now define **weak convergence** as follows. A sequence $\{x_n\}$ is said to converge weakly to x if for every $y \in H$ the sequence of complex numbers $\langle x_n, y \rangle$ converges to $\langle x, y \rangle$. If x_n converges weakly to x we denote it by $x_n \xrightarrow{w} x$.
- (a) Prove that strong convergence implies weak convergence: that is, $x_n \rightarrow x$ implies $x_n \xrightarrow{w} x$.
- (b) Prove that if $x_n \xrightarrow{w} x$ and $\|x_n\| \rightarrow \|x\|$ then $x_n \rightarrow x$.
- (c) Give an example in $\ell^2(\mathbb{N})$ of a sequence which converges weakly but not strongly.
- (3) Let $H = \ell^2(\mathbb{N})$. Fix a positive integer N and define the following linear functional $L(\{\alpha_n\}) = \alpha_N$. Prove that this is a bounded functional, find its norm, and find the element $z_0 \in H$ such that $L(h) = \langle h, z_0 \rangle$ for all $h \in H$.
- (4) Let $H = \ell^2(\mathbb{N} \cup \{0\})$. Prove the following.
- (a) If $\{\alpha_n\} \in H$ then the power series $\sum_{n=0}^{\infty} \alpha_n y^n$ converges for every y with $|y| < 1$. (Hint: use CBS).
- (b) Fix a number a with $|a| < 1$ and define the functional $L_a(\{\alpha_n\}) = \sum_{n=0}^{\infty} \alpha_n a^n$. Find the element $z_0 \in H$ such that $L(h) = \langle h, z_0 \rangle$ for all $h \in H$.
- (c) Find the norm of L_a .
- (5) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq r < p < s \leq \infty$. Prove that if $f \in L_r$ and $f \in L_s$ then $f \in L_p$ and $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$. (Hint: write $p = \lambda r + (1 - \lambda)s$ with $0 < \lambda < 1$, and use Hölder's inequality on suitable powers of f .)