A Differential Geometric Approach to Motion Planning

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Abstract

We propose a general strategy for solving the motion planning problem for real analytic, controllable systems without drift. The procedure starts by computing a control that steers the given initial point to the desired target point for an extended system, in which a number of Lie brackets of the system vector fields are added to the right-hand side. The main point then is to use formal calculations based on the product expansion relative to a P. Hall basis, to produce another control that achieves

*Work supported in part by the National Science Foundation under NSF Grant CDA-8320085.

the desired result on the formal level. It then turns out that this control provides an exact solution of the original problem if the given system is nilpotent. When the system is not nilpotent, one can still produce an iterative algorithm that converges very fast to a solution. Using the theory of feedback nilpotentization, one can find classes of non-nilpotent systems for which the algorithm, in cascade with a precompensator, produces an exact solution in a finite number of steps. We also include results of simulations which illustrate the effectiveness of the procedure.

Introduction

We present a general strategy for solving the Motion Planning Problem (MPP) for controllable systems without drift. Following the line of research initiated by Brockett and Sastry (cf. e.g. [Bro81], [MS90a]), we propose a strategy based on tools from "differential geometric control theory," and making systematic use of Lie brackets of vector fields and Lie algebraic properties.

We consider control systems:

$$\Sigma : \qquad \dot{x} = u_1 f_1(x) + \ldots + u_m f_m(x) \tag{1}$$

where: (1) $f_1, \ldots, f_m$ are *real analytic* vector fields on $\mathbb{R}^n$, and (2) Σ is completely controllable. Notice that the right-hand side of (1) does not contain a term of the form $f_0(x)$, that is, we are assuming that *there is no drift*. It is well known that Condition (2) is equivalent to the LARC (Lie algebra rank condition), i.e. to the condition that $L(f)$ (the Lie algebra of vector fields generated by $f = \{f_1, \ldots, f_m\}$—the *controllability Lie algebra* of Σ) spans $\mathbb{R}^n$ at each point.

The MPP (Motion Planning Problem) is the problem of finding reasonable algorithms producing, for every pair p, q of points, an open-loop control

$$t \longrightarrow u(t) = (u_1(t), \ldots, u_m(t)) \tag{2}$$

that steers p to q.

The controllability condition LARC guarantees that such a control exists for every p and q. Here we want to show a reasonable procedure that will compute one. What is "reasonable" depends on the particular situation and many solutions are possible. Here we describe one solution and present data from simulations.

The MPP has been studied by a number of authors, notably Sastry, Hauser, Murray, Li et al., cf. [HSK89], [MS90b], [MS90a], [SL89]. They have proposed procedures that work very well in a number of special cases. Often, these procedures: (a) attempt to use optimal control; (b) require special assumptions (e.g. that for every k the span of the Lie brackets up to order k has constant dimension); (c) work in special cases.

The strategy proposed here is a modification of the approach mentioned above, in that: (a) it does not in principle require special assumptions on the spans of the Lie brackets: (b) it does not use optimal control; (c) it works exactly for *nilpotent* and *nilpotentizable systems* (defined below); (d) it works approximately for completely general systems, and can be used to produce a "successive approximations" algorithm that converges quite fast to an exact solution.

The Strategy

The main point of the strategy proposed here is to consider, in conjunction with the original system Σ, an *extended system*

$$\Sigma_e : \qquad \dot{x} = v_1 f_1(x) + \ldots + v_m f_m(x) + v_{m+1} f_{m+1}(x) + \ldots + v_r f_r(x) \ , \tag{3}$$

where $f_{m+1}, \ldots, f_r$ are higher-order Lie brackets of the f_i, chosen so that $f_1(x), \ldots, f_r(x)$ span $I\!R^n$ for all x, or at least for all x in some prescribed bounded region $\mathcal{R}$.

Remark 1 *We prefer to allow r to be larger than n for two reasons. First, we can get a better conditioned system for the computation of v as explained below. Second, we can have a fixed set of brackets which span $I\!R^n$ at every point of the region of interest.*

Once we have selected an extended system Σ_e, the solution proposed here of the MPP involves two basic steps:

STEP I: Find a control v that steers p to q for Σ_e.

STEP II: Use v to compute a control u that steers p to q for Σ.

Step I is in principle easy. In fact, to find v, we begin by choosing a C^1 curve $\gamma : [0,T] \longrightarrow I\!R^n$ that goes from p to q. A straight-line segment will work perfectly well, although if we are dealing with, for example, an obstacle avoidance problem, other curves might be more suitable. Having chosen γ, we then write the tangent vector $\dot{\gamma}(t)$ as a linear combination of $f_1(\gamma(t)), \ldots, f_r(\gamma(t))$. The resulting coefficients are the $v_i(t)$. The problem of expressing $\dot{\gamma}(t)$ as a linear combination of $f_1(x), \ldots, f_r(x)$ involves inverting a square matrix (if $r = n$) or computing a pseudoinverse.

Step II is harder and involves some interesting algebra, that will be discussed in detail below.

Some terminology

We recall that a Lie algebra L is said to be *nilpotent* if there is an integer $k > 0$ with the property that all the Lie brackets $[v_1, [v_2, \ldots, [v_k, v_{k+1}] \ldots]]$ vanish. The smallest such k is the *order of nilpotency* of L, and L is said to be *nilpotent of order* k.

The system Σ will be called *nilpotent* if its controllability Lie algebra $L(f)$ is a nilpotent Lie algebra.

We use $A(X_1,\ldots,X_m)$ to denote the algebra of noncommutative polynomials in $(X_1,\ldots,X_m)$. With the Lie bracket defined by $[P,Q]=PQ-QP$, $A(X_1,\ldots,X_m)$ is also a Lie algebra. We then define $L(X_1,\ldots,X_m)$ to be the Lie subalgebra of $A(X_1,\ldots,X_m)$ generated by $X_1,\ldots,X_m$. The elements of $L(X_1,\ldots,X_m)$ are known as *Lie polynomials* in $X_1,\ldots,X_m$. We also use $\hat{A}(X_1,\ldots,X_m)$ and $\hat{L}(X_1,\ldots,X_m)$ to denote, respectively, the set of *noncommutative formal power series* and that of *Lie series* in the X_i. That is, $\hat{A}(X_1,\ldots,X_m)$ consists of all (not necessarily finite) linear combinations $\sum_M c_M M$ of monomials, and $\hat{L}(X_1,\ldots,X_m)$ is the set of those $S\in\hat{A}(X_1,\ldots,X_m)$ such that, for each j, the j-th homogeneous component of S is in $L(X_1,\ldots,X_m)$.

The nilpotent versions $A_k(X_1,\ldots,X_m)$, $L_k(X_1,\ldots,X_m)$ (also denoted A_k^m, L_k^m) are then defined by just killing all the monomials of degree $k+1$. Then A_k^m and L_k^m are known, respectively, as the *free nilpotent associative algebra of order* k and the *free nilpotent Lie algebra of order* k.

Writing down a basis of A_k^m is always very easy, for all one has to do is list all the possible monomials of degree $\leq k$. Writing down a basis of L_k^m is harder because among the brackets there are "relations" (skew-symmetry, Jacobi identity).

The concept of a *P. Hall basis* is a particular way to get around this difficulty and to select a basis. For a definition of a P. Hall basis see [Bat78], [Sus87], or [Sus83]. Here we just give an example.

Example: Let $m=2$ as before, and let the indeterminates be X,Y. Then, up to degree five, the following constitute a P. Hall basis of $L(X,Y)$: X, Y, $[X,Y]$, $[X,[X,Y]]$, $[Y,[X,Y]]$, $[X,[X,[X,Y]]]$, $[Y,[X,[X,Y]]]$, $[Y,[Y,[X,Y]]]$, $[X,[X,[X,[X,Y]]]]$, $[Y,[X,[X,[X,Y]]]]$, $[Y,[Y,[X,[X,Y]]]]$, $[Y,[Y,[Y,[X,Y]]]]$, $[[X,Y],[X,[X,Y]]]$, and $[[X,Y],[Y,[X,Y]]]$.

Step II

It will be important to restrict our choice of the extended system a bit further, by requiring the "new" f_i to be brackets arising from a *P. Hall basis*. Step II then proceeds as follows. We choose k such that all the $f_i, i > m$ have degree $\leq k$. Then:

II. We do ***formal calculations*** on $L_k(X_1, \ldots, X_m)$. The X_i are purely formal ***noncommuting indeterminates***. The calculations are done in two steps:

IIa: Solve a simple differential equation (the "formal extended equation") on the Lie group $G_k(X_1, \ldots, X_m)$ (see below), using P. Hall coordinates.

IIb: Find u from the P. Hall coordinates obtained in IIa.

P. Hall Coordinates

If P is any element of $\hat{A}(X_1, \ldots, X_m)$ or $A_k(X_1, \ldots, X_m)$ that has no constant term, then the exponential e^P is well defined by means of the usual power series. We define

$$\begin{aligned} \hat{G}(X_1, \ldots, X_m) &= \{e^Z : Z \in \hat{L}(X_1, \ldots, X_m)\} , \\ G_k^m = G_k(X_1, \ldots, X_m) &= \{e^Z : Z \in L_k(X_1, \ldots, X_m)\} . \end{aligned} \quad (4)$$

(Examples of elements of $\hat{G}(X, Y)$: $e^X = 1 + X + \frac{1}{2}X^2 + 1/6X^3 + \ldots$, $e^{[X,Y]} = 1 + [X, Y] + \frac{1}{2}[X, Y]^2 + \ldots$, $e^{X-Y+3[X,Y]} = 1 + X - Y + 3[X, Y] + \ldots$. Examples of elements of $G_2(X, Y)$: $e^X = 1 + X + \frac{1}{2}X^2$, $e^{[X,Y]} = 1 + [X, Y]$, $e^{X-Y+3[X,Y]} = 1 + X - Y + 3[X, Y] + \frac{1}{2}(X - Y)^2$. (Notice that the meaning of e^P depends on whether P is regarded as an element of $\hat{G}(X_1, \ldots, X_m)$ or of $G_k(X_1, \ldots, X_m)$.) Then both $\hat{G}(X_1, \ldots, X_m)$ and G_k^m are closed under multiplication, thanks to the Campbell-Hausdorff Formula. Moreover, since $e^{-Z}e^Z = 1$, they are both groups. In fact, G_k^m is the connected simply connected Lie group with Lie algebra L_k^m. We call G_k^m the *free nilpotent Lie group of order k with m infinitesimal generators.*

We now let $B_1, B_2, \ldots, B_s$ be a P. Hall basis of $L_k(X_1, \ldots, X_m)$. Then it is well known that every $S \in G_k(X_1, \ldots, X_m)$ has unique expressions

$$S = e^{h_s B_s} e^{h_{s-1} B_{s-1}} \ldots e^{h_2 B_2} e^{h_1 B_1} , \tag{5}$$

$$S = e^{\tilde{h}_1 B_1} e^{\tilde{h}_2 B_2} \ldots e^{\tilde{h}_{s-1} B_{s-1}} e^{\tilde{h}_s B_s} . \tag{6}$$

The maps $S \longrightarrow (h_1, \ldots, h_s)$ and $S \longrightarrow (\tilde{h}_1, \ldots, \tilde{h}_s)$ are global coordinate charts on G_k^m, and establish global diffeomorphisms between G_k^m and $I\!R^s$. The h_i are the *backward P. Hall coordinates* of S. The $\tilde{h}_i$ are the *forward P. Hall coordinates* of S.

The formal extended equation

We are now ready to give more details about Step IIa. We fix a P. Hall basis of the Lie algebra $L_k(X_1, \ldots, X_m)$, where k is chosen as explained in § . We let E_f be the "evaluation map" that assigns to each $P \in L(X_1, \ldots, X_m)$ the vector field obtained by plugging in the f_i, $i = 1, \ldots, m$, for the corresponding X_i. (For example, $E_f([X_3, [X_1, [X_2, X_3]]]) = [f_3, [f_1, [f_2, f_3]]]$.) We assume that the vector fields $f_{m+1}, \ldots, f_r$ are given by $f_j = E_f(X_j)$ for $j = m + 1, \ldots, r$, where $X_{m+1}, \ldots, X_r$ are such that $X_1, \ldots, X_r$ is a P. Hall basis of L_k^m.

We then solve the O.D.E.

$$\Sigma_{fe}: \qquad \begin{array}{rcl} \dot{S}(t) & = & S(t)\Big(v_1(t)X_1 + \ldots + v_r(t)X_r\Big) , \\ S(0) & = & 1 , \end{array} \tag{7}$$

in backward P. Hall coordinates. (This O.D.E. will be called the "formal extended equation.") That is, we write the solution as a product

$$S(t) = e^{h_s(t) B_s} e^{h_{s-1}(t) B_{s-1}} \ldots e^{h_2(t) B_2} e^{h_1(t) B_1} . \tag{8}$$

The advantage of using backward P. Hall coordinates is that *the functions $h_j(t)$ are easily computed by solving a system of O.D.E.'s with input v.* This system is solved by just successive integrations and algebraic operations. The proof that the computation of the h_j can be done by integrations was given in [Sus86] for the simpler case in which the system Σ_{f_e} is not extended. An almost equally simple proof works for the extended case. (The details will be given in [SL].) The following example illustrates what happens in general.

Example 1. Say $k = 3$, $m = 2$, $r = 4$, $f_3 = [f_1, f_2]$, $f_4 = [f_1, [f_1, f_2]]$. We can take $B_1 = X$, $B_2 = Y$, $B_3 = [X, Y]$, $B_4 = [X, [X, Y]]$, $B_5 = [Y, [X, Y]]$. Then $X_3 = [X, Y]$, $X_4 = [X, [X, Y]]$, $X_5 = [Y, [X, Y]]$. Since $r = 4$, we solve (7) with $v_5 \equiv 0$. Then $h_1(t), \ldots, h_5(t)$ are computed by solving

$$\begin{aligned}
\dot{h}_1 &= v_1 \ , \\
\dot{h}_2 &= v_2 \ , \\
\dot{h}_3 &= h_1 v_2 + v_3 \ , \\
\dot{h}_4 &= \tfrac{1}{2} h_1^2 v_2 + h_1 v_3 + v_4 \ , \\
\dot{h}_5 &= h_2 v_3 + h_1 h_2 v_2 \ , \\
h_1(0) &= h_2(0) \ = \ h_3(0) \ = \ h_4(0) \ = \ h_5(0) = 0 \ .
\end{aligned} \tag{9}$$

Remark 2 *Equation (7) is formally analogous to the equation that defines the "Chen series," introduced in control theory by M. Fliess (cf. [Fli83]), although it differs from it in two ways: (a) in the definition of the Chen series one does not have the extra brackets $X_{m+1} \ldots, X_r$, and (b) (7) is interpreted as evolving in Λ_k^m rather than in the algebra of formal power series.*

Finding u

We now proceed to Step IIb. In this step, we work with the "formal equation"

$$\begin{aligned} \dot{S}^-(t) &= S^-(t)\Big(u_1(t)X_1 + \ldots + u_m(t)X_m\Big) \ , \\ S^-(0) &= 1 \ , \end{aligned} \tag{10}$$

which is the same as the formal extended equation, except that it is *not* extended, i.e. only the indeterminates X_i, $i = 1, \ldots, m$, but no high-order brackets, appear in the right-hand side.

The goal of Step IIb is to solve the following problem:

P *given P. Hall coordinates $h_1, h_2, \ldots, h_s$ of an $S \in G_k(X_1, \ldots, X_m)$, find a control $t \longrightarrow u(t)$ that gives rise to an $S^-(T)$ that has these coordinates.*

Problem **P** can be solved in a number of ways. Here we describe one of several possible solutions. The main idea is to solve separately for each exponential factor, and then concatenate the results. It turns out that this is quite easy to do, if we only ask to do it "modulo factors of higher order." We illustrate this with an example. We switch to forward P. Hall coordinates for ease of exposition. This amounts to a simple algebraic transformation.

Example 2. *Find u for $S(T) = e^{\alpha X}e^{\beta Y}e^{\gamma[X,Y]}e^{\delta[X,[X,Y]]}e^{\epsilon[Y,[X,Y]]}$* .

To solve this, fix once and for all open-loop controls A, B that give rise to e^X, e^Y. Use xA for "x times A," and so on. Use $\#$ for concatenation. (So $A\#B$ means 'A followed by B.')

Then:

- $\alpha A\#\beta B$ gives rise to $e^{\alpha X}e^{\beta Y}$

- $C = \sqrt{\gamma}A\#\sqrt{\gamma}B\#(-\sqrt{\gamma}A)\#(-\sqrt{\gamma}B)$ "almost" gives rise to $e^{\gamma[X,Y]}$. (We are assuming that $\gamma > 0$. If $\gamma < 0$ we just interchange X and Y.) Precisely, C gives rise to $e^{\gamma[X,Y]}e^{\tilde{\gamma}[X[X,Y]]}e^{-\tilde{\gamma}[Y,[X,Y]]}$, where $\tilde{\gamma} = \frac{1}{2}\gamma^{\frac{3}{2}}$.

So far, we have shown that $\alpha A\#\beta Y\#\sqrt{\gamma}A\#\sqrt{\gamma}B\#(-\sqrt{\gamma}A)\#(-\sqrt{\gamma}B)$ gives rise to $e^{\alpha X}e^{\beta Y}e^{\gamma[X,Y]}e^{\tilde{\gamma}[X,[X,Y]]}e^{-\tilde{\gamma}[Y,[X,Y]]}$. What we really want is $e^{\alpha X}e^{\beta Y}e^{\gamma[X,Y]}e^{\delta[X,[X,Y]]}e^{\varepsilon[Y,[X,Y]]}$. To achieve this, it suffices to find a control that gives rise to $e^{\tilde{\delta}[X,[X,Y]]}e^{\tilde{\varepsilon}[Y,[X,Y]]}$, where $\tilde{\delta} = \delta - \tilde{\gamma}$, $\tilde{\varepsilon} = \varepsilon + \tilde{\gamma}$. For this we could use

$$\rho A\#\rho A\#\rho B\#(-\rho A)\#(-\rho B)\#(-\rho A)\#\rho B\#\rho A\#(-\rho B)\#(-\rho A)\#$$
$$\sigma B\#\sigma A\#\sigma B\#(-\sigma A)\#(-\sigma B)\#(-\sigma B)\#\sigma B\#\sigma A\#(-\sigma B)\#(-\sigma A)\ ,$$

where $\rho = (\tilde{\delta})^{\frac{1}{3}}$, $\sigma = (\tilde{\varepsilon})^{\frac{1}{3}}$. If we work in $G_3(X,Y)$ the higher order terms vanish.

The end result is a control u made of 26 pieces that solves our problem exactly. In other words, $S(T)$ is realized in 26 moves.

The main theorems

The following theorems show that the control u computed as above is in fact what we need. Recall that a vector field is *complete* if its integral curves are defined for all times.

Theorem 1 *If $L(f)$ is nilpotent of order k and the vector fields $f_1,\ldots,f_m$ are complete, then the u computed by the above method steers p to q exactly.*

Remark 3 *If the f_i are not complete then it may happen that the trajectory for a given control u is not defined for all t and even if it is defined, the control may fail to steer p to q.*

Theorem 2 *For a general system (1), let $\sigma(p,q,u)$ be the point to which u steers p. Then, on any bounded region $\mathcal{R}$, the error $E(p,q) = ||\sigma(p,q,u) - q||$ satisfies a bound*

$$E(p,q) \leq F(\Delta)\Delta^{\theta} \ , \tag{11}$$

for all $p, q \in \mathcal{R}$, where $\Delta = ||p - q||$, $\theta = 1 + \frac{1}{k}$, and $F : [0,\infty) \longrightarrow [0,\infty]$ is a function which is finite and bounded for Δ near 0.

Remark 4 *The function F is allowed to be infinite if Δ is sufficiently large. This is because in Theorem 2 we are not assuming completeness, so it may in fact happen that the trajectory for u is not even defined for all t.*

Corollary 1 *If $\mathcal{R}$ is a bounded region, then there exists $\Delta > 0$ such that, if $p, q \in \mathcal{R}$ and $||p - q|| \leq \Delta$, then $E(p,q) \leq \frac{1}{2}||p - q||$.*

Definition 1 *The supremum of all the numbers Δ that satisfy the condition of Corollary 1 is called the critical distance for $\mathcal{R}$, and is denoted by $\Delta_c(\mathcal{R})$. Any number $\bar{\Delta}$ such that $0 < \bar{\Delta} \leq \Delta_c(\mathcal{R})$ will be called an admissible step length for $\mathcal{R}$.*

The iterated algorithm

So far, we have explained how to solve the MPP if the system is exactly nilpotent. For a general system, the control u obtained by the prescription of the preceding sections is only an approximate solution. It turns out, however, that we can use our method to obtain a control that steers p to q within any arbitrary prescribed error. This is done by *iterating* the algorithm, and using always distances not greater than the critical distance $\Delta_c(\mathcal{R})$.

The iterated algorithm works as follows. We work in a fixed bounded region $\mathcal{R}$, and fix once and for all an admissible step length $\bar{\Delta}$ for $\mathcal{R}$. Suppose we are given p, q, and an upper bound ε for the permissible error. Then:

1. set $p_0 = p$, $i = 0$, and go to **2**.

2. Apply the algorithm to go from p_i to q_i, where $q_i = p_i + min\left(1, \frac{\bar{\Delta}}{||q-p_i||}\right)(q - p_i)$. Let p_{i+1} be the resulting point.

3. If $||p_{i+1} - q|| \leq \varepsilon$ stop. Else $i := i + 1$ and go to **2**.

We refer to the above as the *iterated algorithm*.

Theorem 8 *Let $\mathcal{R}$ be a bounded region. Let Δ_c be the critical distance for $\mathcal{R}$, and let $\bar{\Delta}$ be such that $0 < \bar{\Delta} \leq \Delta_c$. Assume $||p-q|| = \nu\bar{\Delta}$. Then the iterated algorithm stops in at most $2[\nu] + log_2(\frac{\bar{\Delta}}{\varepsilon})$ steps.*

Feedback Nilpotentization

Recall that a *feedback transformation* of Σ consists of a linear change of controls $u_i = \sum_{j=1}^{m} \beta_{ij}(x)v_j$, such that $\beta(x)$ is a nonsingular matrix for each x, and $\beta(x)$ is smooth as a function of x.

Let us call a system Σ *feedback nilpotentizable* (FN) if it can be made nilpotent by a feedback transformation.

Feedback nilpotentization was studied by H. Hermes, A. Lundell and D. Sullivan in [Her89] and [HLS84]. They proved, for instance, that any system

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x)\ ,\ \ x \in I\!R^3 \qquad (12)$$

for which $f_1(\bar{x}), f_2(\bar{x}), [f_1, f_2](\bar{x})$ are linearly independent is feedback nilpotentizable on a neighborhood of $\bar{x}$. In many cases, the nilpotentization can be carried out explicitly. Examples of several nilpotentizable systems will be shown below.

Results from simulations

We consider first the problem of steering a unicycle (Figure 1).

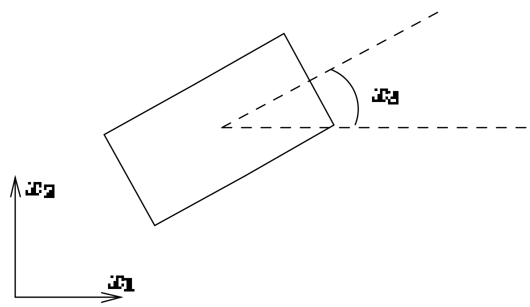


Figure 1: Unicycle.

The controls are the driving speed and the steering speed. The equations for this system are

$$\begin{aligned} \dot{x}_1 &= \cos(x_3)u_1 \ , \\ \dot{x}_2 &= \sin(x_3)u_1 \ , \\ \dot{x}_3 &= u_2 \ . \end{aligned} \tag{13}$$

where (x_1, x_2) are the Cartesian coordinates of the center of the unicycle and x_3 is the angle its main axis makes with the x_1-axis. We can rewrite this system as $\dot{x} = u_1 f_1(x) + u_2 f_2(x)$ where $f_1(x) = (\cos(x_3), \sin(x_3), 0)$ and $f_2(x) = (0, 0, 1)$. Clearly the vectors $f_1(x)$, $f_2(x)$, $[f_1, f_2](x)$ span $I\!R^3$ near $x = 0$, so (13) is locally nilpotentizable as mentioned earlier in § . However, (13) is not nilpotent. In fact, $ad_{f_2}^{2n}(f_1) = (-1)^n(\cos(x_3), \sin(x_3), 0)$ for $n \geq 0$.

We generated first an approximate trajectory for the system using the nilpotent formal system of order 2. In this case $e^{[X,Y]} = e^X e^Y e^{-X} e^{-Y}$, and the formal solution can be written as $S(T) = e^{\alpha_5 X} e^{\alpha_4 Y} e^{\alpha_3 X} e^{\alpha_2 Y} e^{\alpha_1 X}$. We can realize this product of exponentials by choosing a piecewise constant control made up of 5 pieces, each piece generating a move corresponding to a trajectory of either f_1 or f_2. However, the controls need not be piecewise constant. We could use instead controls $u_1(t) = -6\alpha_3 t(t-1)$, $u_2 = 0$ for time 1, etc.. After concatenation we would then get a continuous control. Clearly using higher order polynomials we can obtain as smooth a control as desired.

Specifically, we tried to drive the system from $x_0 = (0,0,0)$ to $x_f = (2,1,0)$. As reference trajectory to compute the v's we used the straight-line segment between the two points parametrized from 0 to 1. The resulting controls are plotted in Figure 2 (bang-bang on top and continuous, piecewise-polynomial on the bottom).

After two iterations the trajectory reaches the target within an error $\varepsilon \leq 0.04$. (The bottom plot represents the unicycle drawn at discrete times along the trajectory.) The backward P. Hall coordinates for our choice of v's are $(-1,0,2)$ for the first iteration and $(-0.16,0,-0.46)$ for the second. (We are using an order 2 approximation.)

This procedure corresponds to guessing that the critical distance is larger than $||x_0 - x_f||$. If, however, we start with a smaller guess $\Delta_g = 0.5$ for the critical distance, then the final trajectory stays closer to the segment from x_0 to x_f but it takes 5 steps to reach the desired target. This is still much smaller than the upper bound of 11 predicted by Theorem 2. Figure 4 shows the result of such a simulation.

Since this system is nilpotentizable we can actually get to the target point in one step exactly. More precisely, this system can be made nilpotent using the following feedback

$$\begin{aligned} u_1 &= \frac{1}{\cos(x_3)} w_1 , \\ u_2 &= \cos^2(x_3) w_2 . \end{aligned} \tag{14}$$

The system then becomes

$$\begin{aligned} \dot{x}_1 &= w_1 , \\ \dot{x}_2 &= \tan(x_3) w_1 , \\ \dot{x}_3 &= \cos^2(x_3) w_2 . \end{aligned} \tag{15}$$

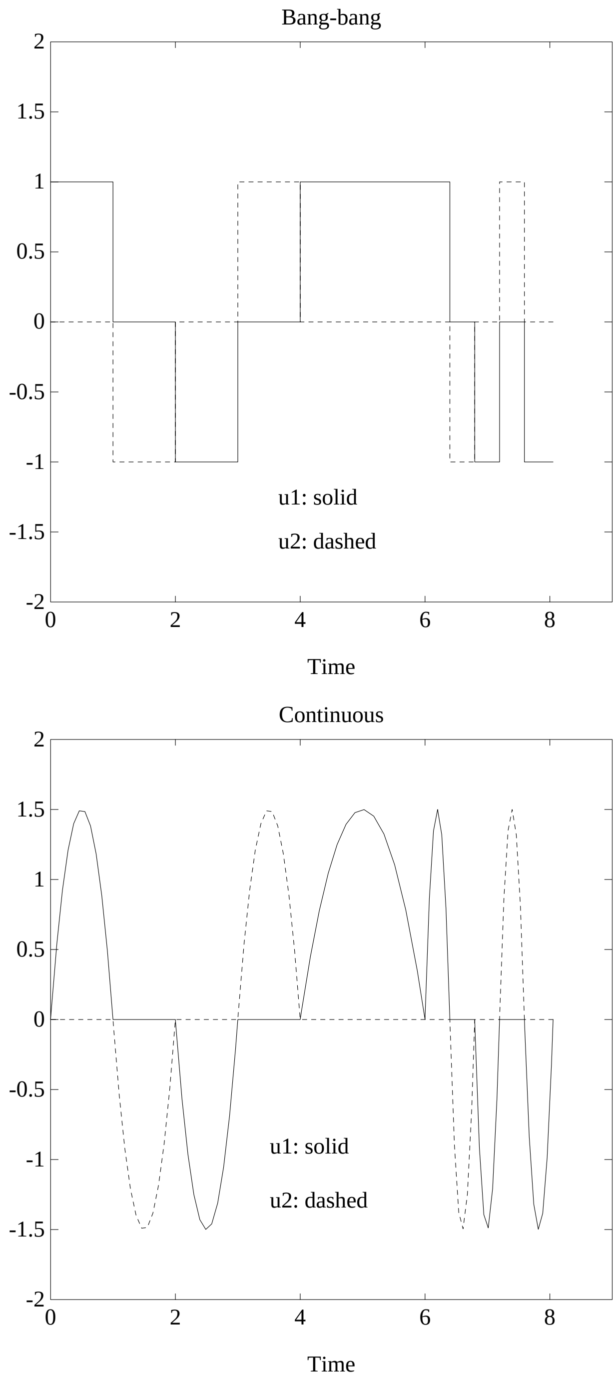


Figure 2: Controls for unicycle.

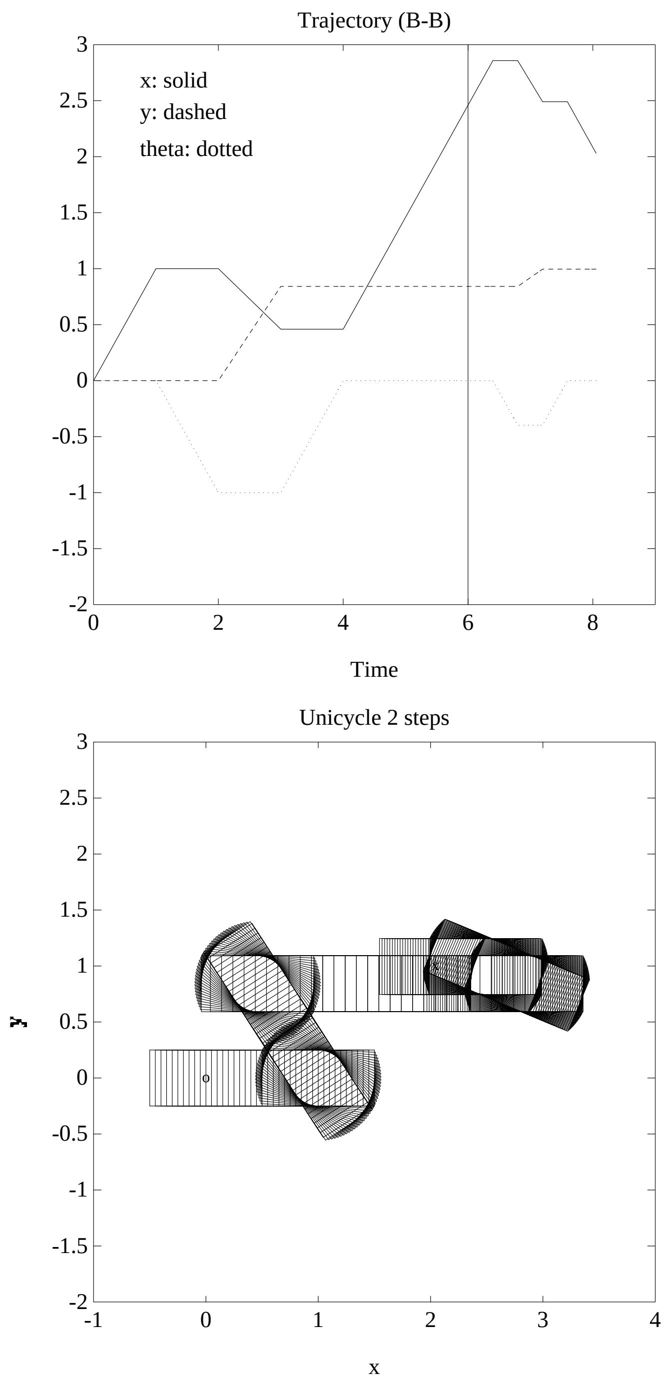


Figure 3: Trajectory generated after 2 iterations with piecewise constant controls.

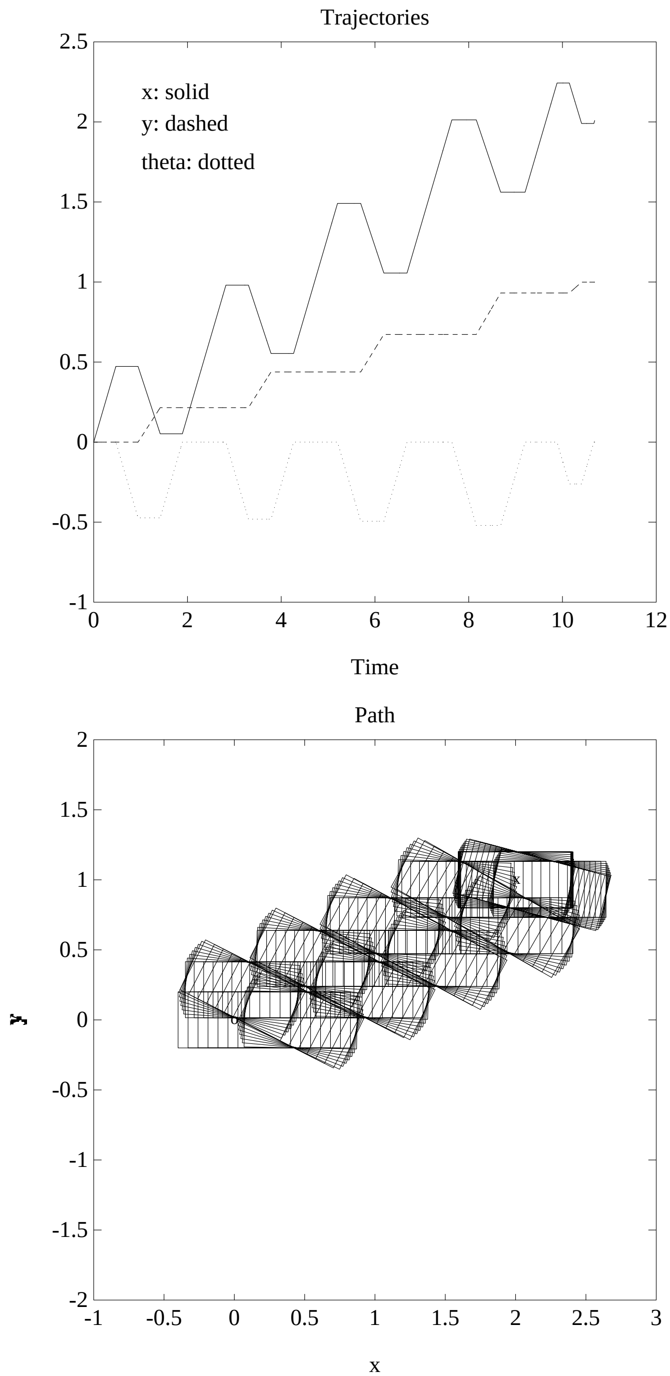


Figure 4: Iterated algorithm with a small critical distance.

which is nilpotent of order 2. We therefore compute the desired controls as follows. First apply the procedure to the nilpotent system to obtain the controls w_i. Then using the feedback (14) compute the controls for the original system. The results of such a simulation (with the same x_0, x_f as before) are presented in Figure 5.

A second example is given by a front wheel drive cart (Figure 6). The controls are the driving speed of the front wheels (u_1) and the turning speed of the front wheels (u_2). The system equations are

$$\begin{aligned} \dot{x}_1 &= \cos(x_3)\cos(x_4)u_1 \ , \\ \dot{x}_2 &= \cos(x_3)\sin(x_4)u_1 \ , \\ \dot{x}_3 &= u_2 \ , \\ \dot{x}_4 &= \frac{1}{l}\sin(x_3)u_1 \ . \end{aligned} \tag{16}$$

where x_1, x_2 are the Cartesian coordinates of the cart, x_3 is the steering angle and x_4 is the angle the main axis of the cart makes with the x_1-axis. Let's rewrite the system as $\dot{x} = u_1 f_1(x) + u_2 f_2(x)$. Then the vectors $f_1(x), f_2(x), [f_1, f_2](x), [f_1, [f_1, f_2]](x)$ span $I\!R^4$ in a neighborhood of $x = 0$. Again we can see that the system is not nilpotent. In fact, $ad_{f_2}^{2n} f_1 = (-1)^n f_1$ for $n \geq 0$. After three iterations an acceptable error (≤ 0.01) is achieved (see Figure 7). (In this plot we rename the variables as $x = x_1$, $y = x_2$, $\psi = x_3$ and $\theta = x_4$.)

The cart is driven from $x_0 = (0,0,0,0)$ to $x_f = (0,-1,0,0)$. The computation of the controls uses the formulas explained in Examples 1 and 2. After some cancellations and regroupings the resulting controls are piecewise constant ($|u_i| \leq 1$) with 19 switches for each step.

This system is also nilpotentizable and therefore a one step procedure can be used to reach the target exactly. The following feedback makes the system nilpotent.

$$\begin{aligned} u_1 &= \frac{1}{\cos(x_3)\cos(x_4)} w_1 \ , \\ u_2 &= \cos^2(x_4)\cos^2(x_3) w_2 - \frac{3}{l}\frac{\sin(x_4)\sin^2(x_3)}{\cos^2(x_4)} w_1 \ . \end{aligned} \tag{17}$$

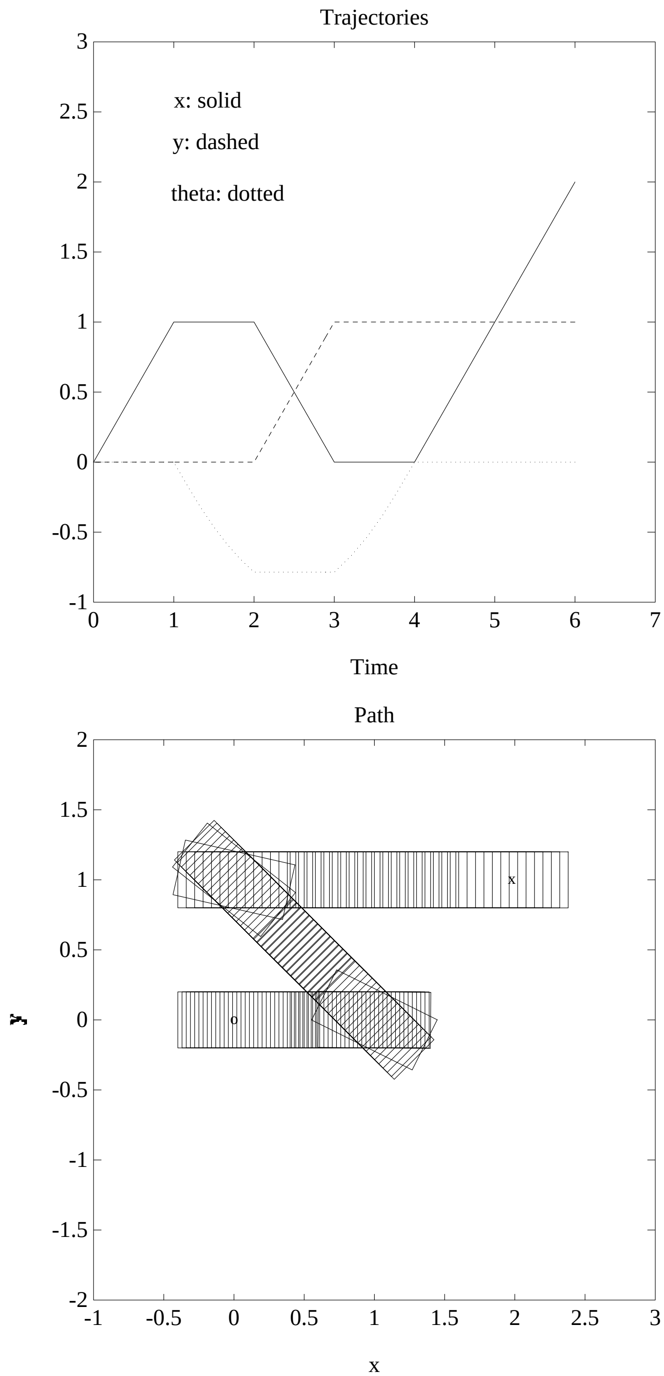


Figure 5: Unicycle with precompensator.

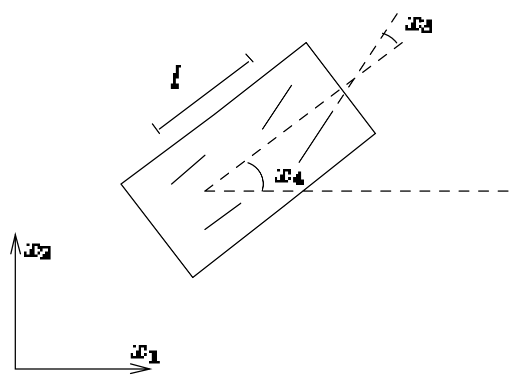


Figure 6: Front wheel drive cart.

In fact, if we make the change of coordinates

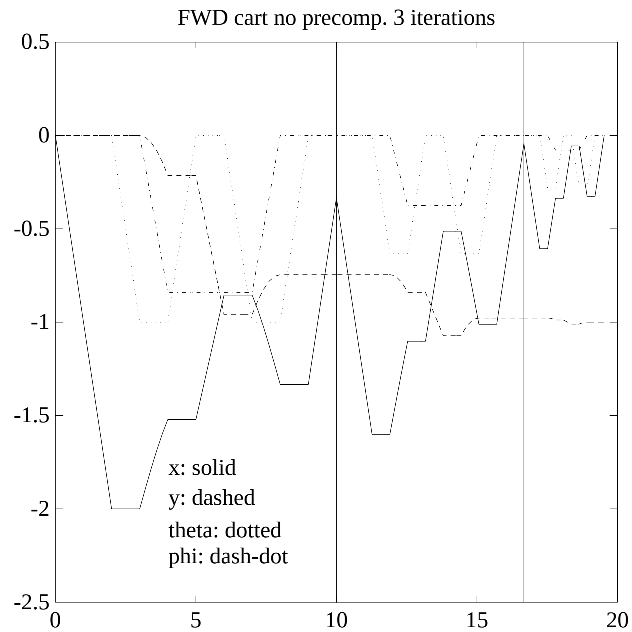
$$z_1 = x_1 \quad z_2 = \frac{\tan(x_3)}{\cos^3(x_4)} \quad z_3 = \tan(x_4) \quad z_4 = x_2 \ , \tag{18}$$

the system becomes,

$$\begin{aligned} \dot{z}_1 &= w_1 \\ \dot{z}_2 &= w_2 \ , \\ \dot{z}_3 &= \frac{1}{l} z_2 w_1 \ , \\ \dot{z}_4 &= z_3 w_1 \ , \end{aligned} \tag{19}$$

which is clearly nilpotent of order 3. Results of the simulation using a precompensator are given in Figure 8.

After the given change of coordinates the end points become $x_0 = (0,0,0,0)$ and $x_f = (0,0,0,-1)$. If f_1, f_2 are the vector fields for the nilpotent system, then the desired motion is exactly in the direction of $[f_1,[f_1,f_2]]$. The backward P. Hall coordinates are in fact $(0,-1,0,0,0)$. In Figure 9 we show plots of each variable against x. Notice that the controls generate periodic trajectories for $\theta = x_4$ and $\psi = x_3$. While θ goes around once, ψ traverses two loops. This is a consequence of the identity $e^{[X,[X,Y]]} = e^X e^{[X,Y]} e^{-X} e^{-[X,Y]}$, which holds in $L_3(X,Y)$, and the fact that the vectors $f_1, f_2, [f_1,f_2], [f_1,[f_1,f_2]]$ correspond to the coordinate axes x_1, x_3, x_4, x_2 respectively. In general, the trajectories prescribed by the forward P. Hall coordinates (which in this example are $(0,0,0,-1,0)$)



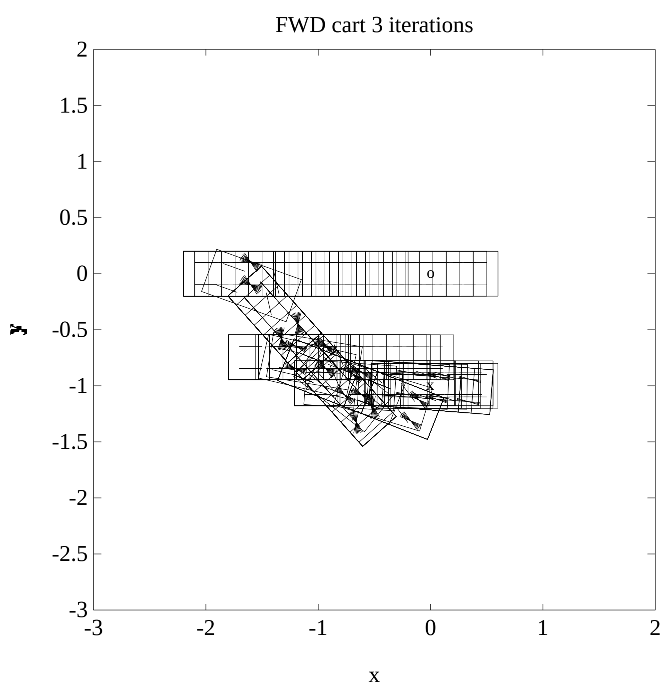
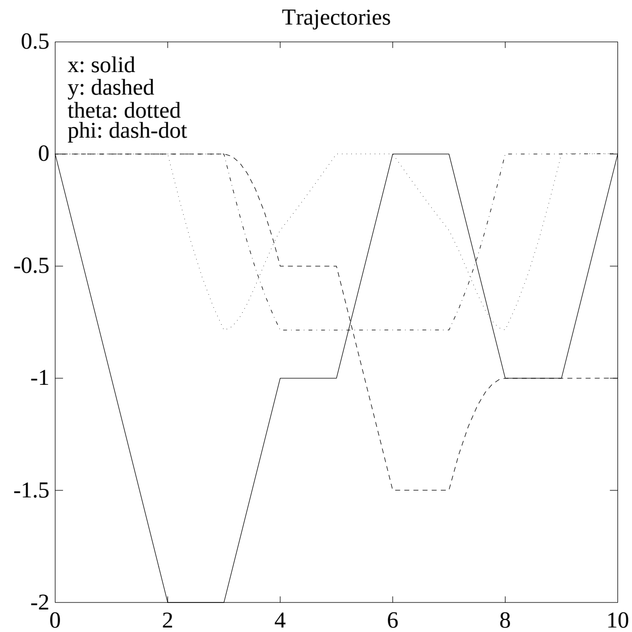


Figure 7: Iterated algorithm applied to front wheel drive cart



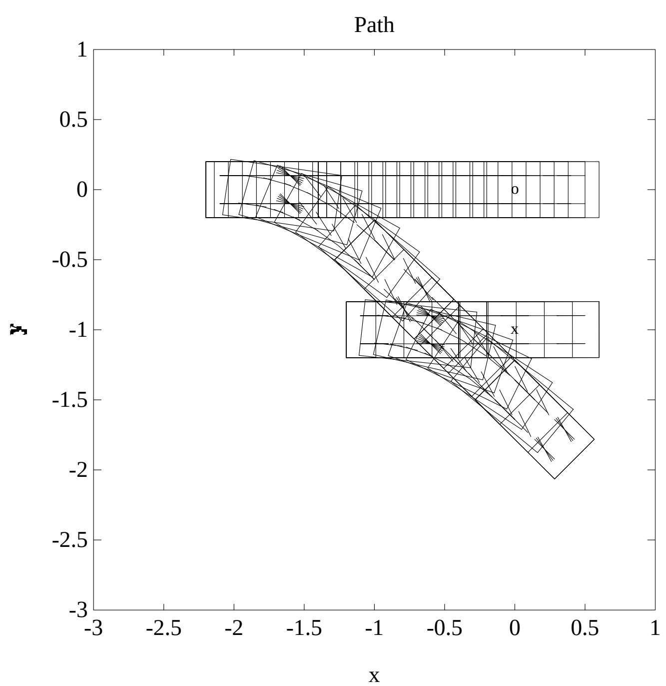


Figure 8: Front wheel drive cart with precompensator.

are made up of moves in the directions of each bracket in the P. Hall basis. The move corresponding to a given bracket results in periodic motions of the variables corresponding to lower order brackets.

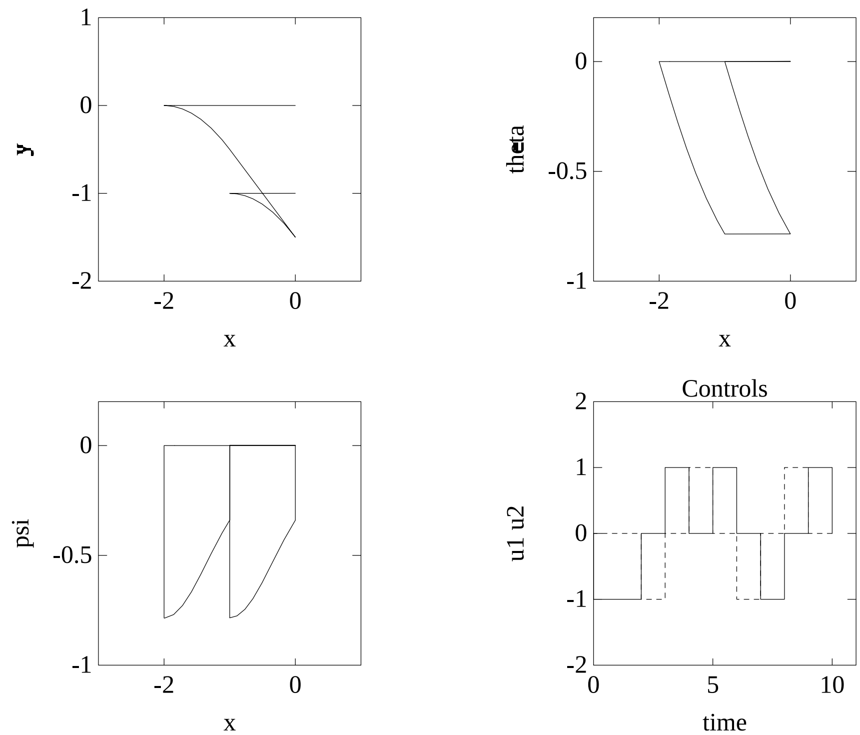


Figure 9: Plots of the different variables against x for the front wheel drive cart with a precompensator

Finally, we present an example of a nilpotentizable system of higher order. Consider a front wheel drive cart as before but which in addition is pulling a trailer (Figure 10). The system has one more equation than before to account for the variable representing the angle the trailer makes with the x_1-axis.

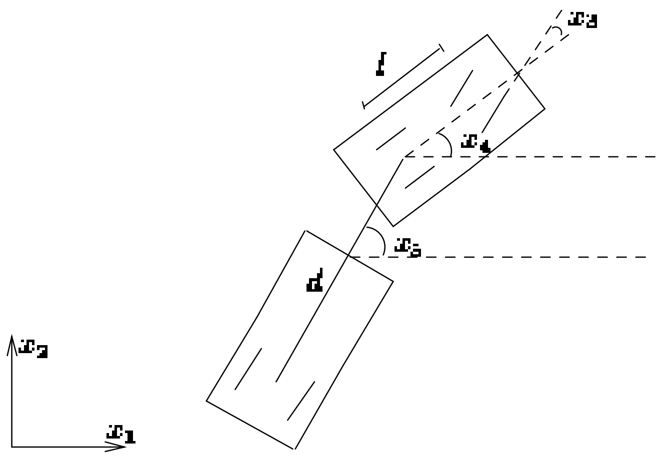


Figure 10: Front wheel drive cart with trailer.

The equations are,

$$\begin{aligned}
\dot{x}_1 &= \cos(x_3)\cos(x_4)u_1 \\
\dot{x}_2 &= \cos(x_3)\sin(x_4)u_1 \\
\dot{x}_3 &= u_2 \ , \\
\dot{x}_4 &= \frac{1}{l}\sin(x_3)u_1 \ , \\
\dot{x}_5 &= \frac{1}{d}\sin(x_4 - x_5)\cos(x_3)u_1 \ .
\end{aligned} \tag{20}$$

The span of $\{f_1(x), f_2(x), [f_1, f_2](x), [f_1, [f_1, f_2]](x), [f_1, [f_1, [f_1, f_2]]](x)\}$ is $I\!R^5$ for x near zero. The system is nilpotentizable. In fact, it is feedback equivalent to a nilpotent system of order 4. The calculations are similar to the front wheel drive cart example but have to be iterated twice (complete details are available in [LS90]). After feedback transformations and change of coordinates the system becomes

$$\begin{aligned}
\dot{x}_1 &= u_1 \ , \\
\dot{x}_2 &= u_2 \ , \\
\dot{x}_3 &= x_2 u_1 \ , \\
\dot{x}_4 &= x_3 u_1 \ , \\
\dot{x}_5 &= \left(x_3\left(\sqrt{1 + x_4^2}\right)^{-1} + x_4\right)u_1 \ .
\end{aligned} \tag{21}$$

Outline of the proofs of Theorems 1 and 2

The quickest way to prove Theorem 1 is to use standard facts about Lie groups, together with a general result proved by R. Palais (cf. Palais [Pal57]):

Theorem 4 *Let L be a finite-dimensional Lie algebra of vector fields on a smooth manifold M, and assume that $f_1, \ldots, f_m$ are generators of L. Assume that the f_i are complete. Let G be the connected simply connected Lie group with Lie algebra L. Then there exists a unique action of G on M whose differential is the identity map.*

We recall that an *action* of a Lie group G on a smooth manifold M is a smooth map $M \times G \ni (x, g) \to xg \in M$ such that $x(gh) = (xg)h$ and $xe = x$ for all $x \in M$, $g, h \in G$, where e is the identity element of G. The *differential* of the action is the map that assigns to each member λ of the Lie algebra L the vector field v_λ defined by

$$v_\lambda(x) = \frac{d}{dt}\bigg|_{t=0} x\,exp(t\lambda)\,, \qquad (22)$$

where "exp" is the exponential map from L to G. In the special case when L is already a Lie algebra of vector fields, it makes sense to talk about the differential being the identity map.

We also recall that it is part of the content of Palais' theorem that, under the stated assumptions, it follows that all the members of L are complete. (It is not true in general that the Lie algebra generated by a set of complete vector fields consists entirely of complete vector fields.)

We can define a map $\nu : L(X_1, \ldots, X_m) \to L(f)$ by just plugging in the f_i for the X_i. Because $L(f)$ is nilpotent of order k, this map is a Lie algebra homomorphism. So the map extends to a Lie group homomorphism (also denoted by ν) from $G_k(X_1, \ldots, X_m)$ to G, where G is the connected simply connected Lie group with Lie algebra $L(f)$. Using Theorem 4, we get an action of G on M. If we are given a control $t \to u(t) = (u_1(t), \ldots, u_m(t))$, $a \le t \le b$, and want to find the corresponding trajectory $t \to x(t)$ starting at $x(a) = \bar{x}$, we can proceed formally by solving the equation $\dot{S}(t) = S(t)(u_1(t)X_1 + \ldots u_m(t)X_m)$ on $G_k(X_1, \ldots, X_m)$ with initial condition $S(a) = 1$ and then "projecting down" to M. Precisely, we can define $x(t) = \bar{x}\,\nu(S(t))$. If $Y \in L(X_1, \ldots, X_m)$ is arbitrary, then $\nu\Big(S(t)e^{sY}\Big) = \nu\Big(S(t)\Big)\,\nu(e^{sY}) = \nu\Big(S(t)\Big)\,e^{s\nu(Y)}$, because ν is a homomorphism. Then But

$$\frac{d}{ds}\bigg|_{s=0} \nu\Big(S(t)e^{sY}\Big) = \frac{d}{ds}\bigg|_{s=0} \nu\Big(\Gamma(s)\Big)$$

if Γ is any curve in $G_k(X_1, \ldots, X_m)$ such that $\Gamma(0) = S(t)$ and $\dot{\Gamma}(0) = S(t)Y$. In particular, we can take $\Gamma(s) = S(t+s)$, in which case $Y = \sum_i u_i(t)X_i$, and we get

$$\frac{d}{d\tau}\bigg|_{\tau=t} \Big(\bar{x}\,\nu\Big(S(\tau)\Big)\Big) = \nu\Big(\sum_i u_i(t)X_i\Big)\Big(\bar{x}\,\nu\Big(S(t)\Big)\Big)\ ,$$

i.e. $\dot{x}(t) = \sum_i u_i(t) f_i(x(t))$. So $x(\cdot)$ is a trajectory, as stated.

It is clear that a similar conclusion holds for trajectories of the extended system. Using this, the assertion of Theorem 1 follows immediately: if u and v give rise to the same $S(T)$, then for any given initial condition $\bar{x}$ the trajectories corresponding to u and v will go through $\bar{x}\,\nu(S(T))$ at time T, i.e. both u and v steer $\bar{x}$ to the same point. ∎

In order to prove Theorem 2, it is important to use a convenient formalism. It turns out that it is best to use the formalism in which vector fields and flows act on the right, as suggested by Agrachev, Gamkrelidze and Sarychev in their papers on the "chronological calculus" (cf., e.g. [AG78]).

Also, since the possibility of explosions introduces extra complications, we prefer to do most of the work with vector fields in $\mathcal{D}^n(I\!R^n)$, the space of infinitely differentiable vector fields on $I\!R^n$ with compact support, and then reduce the general case to this one by means of a cutoff argument.

The formalism is as follows. Points of $I\!R^n$, as well as many other objects such as tangent vectors at a point, can be regarded as Schwartz distributions, i.e. as linear functionals on the space $\mathcal{D}(I\!R^n)$ of compactly supported real-valued functions of class C^∞ on $I\!R^n$. Let $Diff(I\!R^n)$ be the set of all globally defined surjective diffeomorphisms $\Phi : I\!R^n \to I\!R^n$.

For $\varphi \in \mathcal{D}(I\!R^n)$, $f \in \mathcal{D}^n(I\!R^n)$, or $\Phi \in Diff(I\!R^n)$, let us use $x\varphi$, xf, $x\Phi$ as alternative notations for $\varphi(x)$, $f(x)$, $\Phi(x)$. If v is a tangent vector at x, and $\varphi \in \mathcal{D}(I\!R^n)$, then $v\varphi$ denotes what is usually referred to as the "directional derivative of φ at x in the direction of v," i.e. $\langle v, \nabla\varphi(x)\rangle$. If $v = \lim_{h\to 0} \frac{x(h)-x}{h}$, where $x(\cdot)$ is a C^1 curve such that $x(0) = x$, then $v\varphi$ is just equal to $\lim_{h\to 0} \frac{x(h)\varphi - x\varphi}{h}$. Every $\Phi \in Diff(I\!R^n)$ acts on $\mathcal{D}(I\!R^n)$ by an obvious duality: we let $\Phi\varphi(x) = \varphi(\Phi(x))$ i.e., in our notation, $x(\Phi\varphi) = (x\Phi)\varphi$. (So from now on we can just write $x\Phi\varphi$.) On the other hand, this same duality now enables us to have Φ act on the right on tangent vectors: if v is a tangent vector at x, then $v\Phi$ is the tangent vector at $x\Phi$ defined as the map $\varphi \to v(\Phi\varphi)$, so that $(v\Phi)\varphi = v(\Phi\varphi)$ and, once again, we can drop the parenthesis and write $v\Phi\varphi$. Notice that $v\Phi$ is the vector that in Differential Geometry is usually denoted $\Phi_* v$, or $d\Phi(v)$, i.e. the image of v under the differential of Φ. If we write maps $\mu : I\!R^n \to I\!R^n$ (such as members of $Diff(I\!R^n)$ or of $\mathcal{D}^n(I\!R^n)$) as columns of real-valued functions, and use $D\mu$ to denote the Jacobian matrix of μ (i.e. the square matrix of functions whose rows are the gradients of the components of f), then $v\Phi$ is just $D\Phi(x) \cdot v$.

The space $\mathcal{D}^n(I\!R^n)$ is a Lie algebra. Moreover, for any $f \in \mathcal{D}^n(I\!R^n)$ we can define the norms

$$\|f\|_0 = \sup\{\|f(x)\| : x \in I\!R^n\} , \tag{23}$$

$$\|f\|_1 = \sup\{\|Df(x)\| : x \in I\!R^n\} . \tag{24}$$

For $f \in \mathcal{D}^n(I\!R^n)$, integral trajectories are defined for all times. We let $t \to xe^{tf}$ denote the integral curve of f that goes through x at time $t = 0$. It is clear that the map $(t, x) \to xe^{tf}$ is of class C^∞, and satisfies $xe^{(t+s)f} = xe^{tf}e^{sf}$ for all t, s, and

$$\frac{d}{dt} xe^{tf} = xe^{tf} f . \tag{25}$$

The maps e^{tf} belong to $Diff(I\!R^n)$. In particular, $xe^{tf}ge^{-tf}$ is a well defined tangent vector at x whenever $g \in \mathcal{D}^n(I\!R^n)$, so $e^{tf}ge^{-tf}$ is a well defined vector field, which is easily seen to be in $\mathcal{D}^n(I\!R^n)$. This vector field will be denoted $e^{t\,\mathrm{ad}_f}(g)$. If we fix x, f, g, t, and let $v(t) = xe^{t\,\mathrm{ad}_f}(g)$, then $v(t) = w(0)$, where $s \to w(s)$ is the solution of the variational equation

$$\dot{w}(s) = Df(xe^{sf}) \cdot w(s) \quad , \tag{26}$$

with terminal condition $w(t) = g(xe^{tf})$. This implies in particular that the bound $||w(t)|| \leq e^{|t|\,||f||_1}||g(xe^{tf})||$ holds. So

$$||e^{t\,\mathrm{ad}_f}(g)||_0 \leq e^{|t|\,||f||_1}||g||_0 \,. \tag{27}$$

We also need an estimate on the Lipschitz norm of e^{tf}. Define

$$||\Phi||_{Lip} = \sup\left\{\frac{||x\Phi - y\Phi||}{||x-y||} : x, y \in I\!R^n \,,\, x \neq y\right\} \,. \tag{28}$$

Then Gronwall's inequality implies that

$$||e^{tf}||_{Lip} \leq e^{|t|\,||f||_1} \quad . \tag{29}$$

Vector fields act as differential operators on functions: if $f \in \mathcal{D}^n(I\!R^n)$ and $\varphi \in \mathcal{D}(I\!R^n)$, then $f\varphi$ is the function $x \to (xf)\varphi$. (Recall that xf is a tangent vector, and tangent vectors act on functions.) Once again, the definitions are set up so that $x(f\varphi) = (xf)\varphi$, so we can simply write $xf\varphi$. The function $f\varphi$ is often denoted $L_f\varphi$ in the control literature, and referred to as the "Lie derivative of φ in the direction of f," but the notation used here is both more convenient and consistent with the one commonly used in Differential Geometry, where vector fields are *defined* as differential operators on functions.

If we let $DO(I\!R^n)$ be the algebra of smooth linear differential operators on $I\!R^n$ (so that every member of $DO(I\!R^n)$ acts as a linear operator from $\mathcal{D}(I\!R^n)$ to $\mathcal{D}(I\!R^n)$), then products of vector fields are well defined members of $DO(I\!R^n)$.

It is then clear that

$$\begin{aligned} \frac{d}{dt}e^{t\,ad_f}(g) &= e^{tf}(fg-gf)e^{-tf} \\ &= e^{t\,ad_f}([f,g]) \ . \end{aligned} \tag{30}$$

From this one gets by successive integrations by parts

$$e^{t\,ad_f}(g) = \sum_{j=0}^{k} \frac{t^j}{j!} ad_f^j(g) + \int_0^t \frac{(t-s)^k}{k!} e^{s\,ad_f}\left(ad_f^{k+1}(g)\right) ds \ . \tag{31}$$

We now assume that we are given vector fields $g_1, \ldots, g_m \in \mathcal{D}^n(I\!R^n)$, and an integer $k > 0$. We let $B_1 = X_1, \ldots, B_m = X_m$, and then choose $B_{m+1}, \ldots, B_s$ so that $B_1, \ldots, B_s$ is a P. Hall basis of $L_k(X_1, \ldots, X_m)$. We let $g_j = E_g(B_j)$ for $j = m+1, \ldots, s$, where E_g is the "plugging in the g's" evaluation map, i.e. the map $L_k(X_1, \ldots, X_m) \to \mathcal{D}^n(I\!R^n)$ defined by plugging in $g_1, \ldots, g_m$ for $X_1, \ldots, X_m$. (Notice that this map is *not* in general a Lie algebra homomorphism. For instance, if $k = 1$, then $[X_1, X_2] = 0$ in $L_k(X_1, \ldots, X_m)$, but $[g_1, g_2]$ need not vanish.)

We now consider a curve $t \to x(t), 0 \le t \le 1$, which is a solution of a differential equation

$$\dot{x}(t) = w_\nu(t)g_\nu(x(t)) + \ldots + w_s(t)g_s(x(t)) + h_t(x(t)) \ , \tag{32}$$

where we assume that $1 \le \nu \le s$, each control $t \to w_j(t)$ is bounded and measurable, and $t \to h_t$ is a measurable bounded $\mathcal{D}^n(I\!R^n)$-valued function. (I.e. $h_t(x)$ is jointly measurable and bounded as a function of t and x, and $h_t \in \mathcal{D}^n(I\!R^n)$ for each t.) We rewrite this in our notation as

$$\dot{x}(t) = x(t)\Big(w_\nu(t)g_\nu + \ldots + w_s(t)g_s + h_t\Big) \ . \tag{33}$$

We let $W_\nu(t) = \int_0^t w_\nu(\tau)d\tau$, and write

$$y(t) = x(t)e^{-W_\nu(t)g_\nu} \ . \tag{34}$$

An easy calculation then shows that $y(t)$ satisfies

$$\dot{y}(t) = y(t)e^{W_\nu(t)ad_{g_\nu}}\Big(w_{\nu+1}(t)g_{\nu+1}(x(t)) + \ldots + w_s(t)g_s(x(t)) + h_t(x(t))\Big) \ . \tag{35}$$

We then write each vector field $e^{W_\nu(t)ad_{g_\nu}}(g_\mu)$, $\nu+1\le\mu\le s$, as

$$e^{W_\nu(t)ad_{g_\nu}}(g_\mu)=\sum_{j=0}^{\kappa_\mu}\frac{W_\nu(t)^j}{j!}ad_{g_\nu}^j(g_\mu)+\int_0^{W_\nu(t)}\frac{(W_\nu(t)-\omega)^{\kappa_\mu}}{\kappa_\mu!}e^{\omega\, ad_{g_\nu}}\Big(ad_{g_\nu}^{\kappa_\mu+1}(g_\mu)\Big)d\omega\ , \tag{36}$$

using (31). The κ_μ are chosen so that $ad_{g_\nu}^{\kappa_\mu}(g_\mu)$ has degree $\le k$ but $ad_{g_\nu}^{\kappa_\mu+1}(g_\mu)$ has degree $>k$. The brackets $ad_{g_\nu}^j(g_\mu)$, $j\le\kappa_\mu$, can be written as linear combinations $\sum_\alpha c_{j\mu\alpha}g_\alpha$ of $g_1,\ldots,g_s$, with constant coefficients. (This follows because it is true on the purely formal level, since the B_i form a basis of $L_k(X_1,\ldots,X_m)$, and therefore the $ad_{X_\nu}^j(X_\mu)$ are linear combinations of them.)

At this point it is crucial to observe that in these expressions all the $c_{j\mu\alpha}$ vanish for $\alpha\le\nu$. To see this observe that the elements of the P. Hall basis are ***homogeneous***. Therefore in the expression of a homogeneous bracket B as a linear combination of the B_i only B_i's of the same degree as B will occur. The brackets $ad_{g_\nu}^j(g_\mu)$ with $j>0$ all have degree strictly larger than the degree of g_ν, and therefore also strictly larger than the degrees of the g_η with $\eta<\nu$. (Because in the P. Hall basis the ordering is compatible with degree, i.e. $degree(B_i)\le degree(B_{i'})$ if $i<i'$.) So in those brackets $g_1,\ldots,g_\nu$ do not occur. As for the $ad_{g_\nu}^j(g_\mu)$ with $j=0$, these are, of course, the g_μ themselves, for $\mu>\nu$, so again their expressions do not involve $g_1,\ldots,g_\nu$. Therefore y satisfies

$$\dot y(t)=y(t)\Big(\tilde w_{\nu+1}(t)g_{\nu+1}+\ldots+\tilde w_s(t)g_s+\tilde h_t\Big)\ , \tag{37}$$

with obvious formulae for the $\tilde w_j$ and $\tilde h$:

$$\tilde w_\mu=\sum_{\rho,j}\frac{W_\nu^j}{j!}w_\rho c_{j\rho\mu}\ , \tag{38}$$

$$\tilde h_t=\sum_\rho w_\rho(t)\int_0^{W_\nu(t)}\frac{(W_\nu(t)-\omega)^{\kappa_\rho}}{\kappa_\rho!}e^{\omega\, ad_{g_\nu}}\Big(ad_{g_\nu}^{\kappa_\rho+1}(g_\rho)\Big)d\omega+e^{W_\nu(t)ad_{g_\nu}}(h_t)\ . \tag{39}$$

We want to use these formulas to estimate $\tilde w$. It turns out that the best form of the estimates is in terms of the ***dilation norm***. Let δ_i be the degree of B_i. Define $|\ldots|:I\!R^s\to I\!R^+$ by

$$|(w_1,\ldots,w_s)|=\sum_{i=1}^s|w_i|^{\frac{1}{\delta_i}}\ . \tag{40}$$

Then $|\ldots|$ is called the "dilation norm" (even though it is not a norm in the usual sense). For a control $t \to w(t) \in I\!R^s$ we define the dilation norm $|w(\cdot)| = \sup\{|w(t)| : 0 \le t \le 1\}$. It then follows from (38) that $|\tilde{w}(\cdot)| \le K|w(\cdot)|$, where K is a purely combinatorial constant, depending only on m, k, and the choice of a P. Hall basis.

We also estimate $\tilde{h}$. Using (39) and (27) we conclude that

$$||\tilde{h}_t||_0 \le e^{|w(\cdot)|^{\delta_\nu}||g_\nu||}\left(||h_t||_0 + \hat{K}|w|^{(\kappa_\rho+1)\delta_\nu+\delta_\rho}\right), \tag{41}$$

where $\hat{K}$ is a constant that depends only on the $||\ldots||_0$ norms of the "bad brackets" (i.e. the brackets $ad_{g_\nu}^{j+1}(g_\rho)$ with $j\delta_\nu + \delta_\rho \le k < (j+1)\delta_\nu + \delta_\rho$).

Now let $g_1, \ldots, g_m \in \mathcal{D}^n(I\!R^n)$, k, and the brackets $g_{m+1}, \ldots, g_s$ be as above, and such that in addition $g_1(x), \ldots, g_s(x)$ span $I\!R^n$ for all x such that $||x|| \le R + 1$. It then follows in particular that, if $v \in I\!R^n$ is any vector, and $||x|| \le R + 1$, then we can write $v = \sum_i v_i g_i(x)$ with the v_i such that $|v_i| \le C||v||$ for some fixed constant C. If we now pick two points p, q in $I\!R^n$, and take $\gamma : [0,1] \to I\!R^n$ to be defined by $\gamma(t) = p + t(q - p)$, we can then define the functions $t \to v_i(t)$ as before, and get the bound $|v_i(t)| \le C||\dot{\gamma}(t)||$, i.e. $|v_i(t)| \le C\Delta$, where $\Delta = ||q - p||$. In particular, this implies that the dilation norm of $v(\cdot)$ is bounded by a fixed constant times $\Delta^{\frac{1}{k}}$. (From now on we only consider points p, q such that $\Delta \le 1$, so that $\Delta \le \Delta^\sigma$ if $0 < \sigma < 1$.)

We then compute u from v as explained in §9. We now have $T = 1$. We observe that the P. Hall coordinates of $S(T)$ are bounded by a fixed constant times Δ, and therefore the u computed by our method is bounded by a constant times $\Delta^{\frac{1}{k}}$. Since u only has components associated to the B_i of degree 1, this implies that the dilation norm of u is also bounded by a constant times $\Delta^{\frac{1}{k}}$.

We now consider the trajectories $t \to \hat{x}(t)$, $t \to \check{x}(t)$ of $\dot{x} = x(w_1 g_1 + \ldots + w_s g_s)$ with initial condition p and for the two choices $w = v$ and $w = u$. In both cases, we perform repeatedly the transformation described above, obtaining trajectories $t \to \hat{x}_j(t)$, $t \to \check{x}_j(t)$ of $\dot{x} = x(v_j^j g_j + \ldots + v_s^j g_s + \hat{h}_t^j)$ and $\dot{x} = x(u_j^j g_j + \ldots + w_s^j g_s + \check{h}_t^j)$, respectively. (Naturally, $\hat{x}_1 \equiv \hat{x}$, $\check{x}_1 \equiv \check{x}$, $\hat{h}_t = 0$, $\check{h}_t = 0$, $v^1 = v$, $u^1 = u$.) After s applications of these transformations, we end up with trajectories $\hat{x}_{s+1}$, $\check{x}_{s+1}$ of the systems $\dot{x} = x\hat{h}_t^{s+1}$, $\dot{x} = x\check{h}_t^{s+1}$. By repeated applications of our estimates we get $||\hat{h}_t||_0 \leq F(\Delta)\Delta^{1+\frac{1}{k}}$, where F is a function that remains bounded as $\Delta \to 0$, and a similar bound for $||\check{h}_t||_0$. From this it follows that $||\hat{x}_{s+1} - p||$ and $||\check{x}_{s+1} - p||$ are bounded by $F(\Delta)\Delta^{1+\frac{1}{k}}$, and so $||\hat{x}_{s+1} - \check{x}_{s+1}|| \leq 2F(\Delta)\Delta^{1+\frac{1}{k}}$.

Now, while we apply the transformations to the $\hat{x}_j$, $\check{x}_j$, we can simultaneously apply them as well to the solutions $\hat{S}_j$, $\check{S}_j$ of the corresponding formal equations, beginning with $\hat{S}(t)$ and $\check{S}(t)$, the solutions of the formal extended equation $\dot{S} = S(v_1X_1 + \ldots + v_sX_s)$, and of the formal equation $\dot{S} = S(u_1X_1 + \ldots + u_mX_m)$, respectively. It is easily seen that the formulas that give the new v's and u's in terms of the old ones are exactly the same for the formal systems as for the original differential equation. However, in the formal case, after we get through all the transformations, the right-hand sides of both equations are just zero, because the brackets of degree $> k$ vanish.

So we end up with the following equations:

$$\begin{aligned}
\hat{x}(1) &= \hat{x}_{s+1}(1)e^{\alpha_s g_s} \ldots e^{\alpha_1 g_1}\,, \\
\check{x}(1) &= \check{x}_{s+1}(1)e^{\beta_s g_s} \ldots e^{\beta_1 g_1}\,, \\
\hat{S}(1) &= e^{\bar{\alpha}_s X_s} \ldots e^{\bar{\alpha}_1 X_1}\,, \\
\check{S}(1) &= e^{\bar{\beta}_s X_s} \ldots e^{\bar{\beta}_1 X_1}\,.
\end{aligned} \tag{42}$$

Because of the parallelism pointed out above between the transformations for the formal and non-formal systems, we have $\alpha_i = \bar{\alpha}_i$ and $\beta_i = \bar{\beta}_i$ for all i. On the other hand, u is constructed so that $\hat{S}(1) = \check{S}(1)$, so that (by the uniqueness of the P. Hall coordinates) $\alpha_i = \beta_i$ as well. We then have $\hat{x}(1) - \check{x}(1) = \hat{x}_{s+1}(1)\Phi - \check{x}_{s+1}(1)\Phi$, where Φ is the map

$$z \to ze^{\alpha_s g_s} \ldots e^{\alpha_1 g_1}\,. \tag{43}$$

In view of (29), the Lipschitz norm of Φ is bounded by e^{CK}, where K is a bound for the norms $||g_j||_1$ and C is a bound for the α_j. But then C can be taken to be bounded by a constant times Δ. So we have the bound

$$\begin{aligned} ||\hat{x}(1) - \check{x}(1)|| &\leq e^{K'\Delta} ||\hat{x}_{s+1}(1) - \check{x}_{s+1}(1)|| \\ &\leq F(\Delta)\Delta^{1+\frac{1}{k}} \end{aligned} \tag{44}$$

for same choice of F.

This completes the proof for the case of vector fields in $g_i \in \mathcal{D}^n(I\!R^n)$. If we now have vector fields f_i that are not necessarily of compact support, we proceed as follows. Pick a ball $B_R = \{x : ||x|| \leq R\}$. Let $g_i = \psi f_i$, where ψ is a function in $\mathcal{D}(I\!R^n)$ which is equal to 1 for $||x|| \leq R+1$. Pick p and q in the ball B_R. Then the f_i and the g_i agree on the segment from p to q, so the v_i are the same whether we compute them using the f_i or the g_i. The u_i are therefore the same as well, since the calculation of u from v is purely formal. In particular, this implies that the u_i are bounded by a fixed constant times $\Delta^{\frac{1}{k}}$. Therefore, if Δ is sufficiently small, the trajectory of the original system that corresponds to u cannot leave the ball $\{x : ||x|| \leq R+1\}$. But then this trajectory is also a trajectory of the g-system, and the bound proved above holds. ∎

Conclusion

The method presented above for solving the MPP rests on a sound theoretical basis, and our preliminary simulation results appear to show that it performs rather well in practice. Naturally, the procedure we have outlined can still be improved in a number of ways, and this requires continued research on several important issues.

A crucial problem is that of estimating the critical distance. The proof of Theorem 2 only provides a very rough estimate of $\Delta_c(\mathcal{R})$. In the simulations, $\Delta_c(\mathcal{R})$ appears to be much larger.

Another important question is that of the class of controls u being used. The method described here involves the concatenation of controls that realize motion along the trajectories of the vector fields f_i. One can use bang-bang controls, but this is not necessary.

For instance, the role of A in Example .1 can be played by a constant control $u_1 \equiv 1$, $u_2 \equiv 0$, during time 1, but we can also use instead a nonconstant control —e.g. $u_1(t) = \frac{1}{2} sin\, t$, $0 \leq t \leq \pi$ — as long as its integral is equal to 1. In this way we end up with continuous controls. Alternatively, we can use more complicated pieces to build up our controls—e.g. controls that directly generate exponentials $e^{\alpha X + \beta Y}$. In this way, the total number of pieces can be cut down considerably.

A third important issue is that of fully exploiting the possibilities of feedback nilpotentization. This requires that one look for new classes of nilpotentizable systems, and also that one improve the existing nilpotentization results by making them as explicit as possible.

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