Discontinuous Stabilizing Feedback Using Partially Defined Lyapunov Functions

G.A. Lafferriere
Department of Mathematics
Portland State University
Portland, OR 97207-0751

Abstract

We generalize earlier results on the construction of discontinuous feedback laws from smooth but partially defined control Lyapunov functions. The resulting feedback law is continuous at the origin and smooth except on a hypersurface of codimension 1. We provide a formula for the feedback law which is in a sense "universal." The new results presented cover situations where trajectories of the closed loop system switch an infinite number of times between regions where smooth control Lyapunov functions exist. The conditions on the system vector fields can be verified without solving the differential equations and are therefore in the spirit of the "direct" methods of Lyapunov. Using a recently developed formula we are also able to guarantee certain bounds on the feedback controls provided that the Lyapunov property can be satisfied using controls values in the unit ball.

1. Introduction

We study systems for which there are smooth Lyapunov-type functions which are only partially defined. We give conditions under which these functions can be "pasted together" to effectively guarantee global asymptotic stability. We only consider the situation with overlapping regions. In this case the verification of the Lyapunov property can be done using gradients, as in the smooth case (see [8] and [9] for other approaches).

In [1], Artstein guarantees the existence of a globally stabilizing feedback law provided the system has a smooth *control Lyapunov function*. Sontag [7] gave a constructive proof providing a formula for the feedback law in terms of Lie derivatives of the Lyapunov function and which is in a sense "universal." Sontag and Lin [11] then used a similar idea to allow for certain types of control constraints. Here we generalize those ideas to certain cases when we only have a *piecewise smooth* Lyapunov function.

The need for the study of such discontinuous feedback arises specially in systems where no continuous static feedback is possible. Such is the case for the class of non-holonomic systems of the form $\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$ (which appear naturally in the study of mechanical systems). These systems can not be stabilized by continuous static feedback at the origin since they do not satisfy Brockett's necessary conditions (see e.g. [10], Section 4.8). Stabilizing feedback laws have been found for several such systems using either time-varying feedback or dynamic feedback; see [3] and references there. Recently, however, piecewise continuous feedback laws were presented for two such examples in [2] and [5].

Here we construct a piecewise continuous globally stabilizing feedback law from partially defined smooth control Lyapunov functions. Using a different "universal" formula we can also obtain bounded feedback controls whenever the Lyapunov property can be satisfied with suitably bounded controls.

2. Definitions

Notation. We denote by $\overline{M}$ the closure of the set M. If g is a vector field on $\mathbf{R}^n$ and $V : \mathbf{R}^n \longrightarrow \mathbf{R}$ is a smooth function then we denote by L_gV the Lie derivative of V with respect to g (i.e. $L_gV(x) = \nabla V(x) \cdot g(x)$).

Definition 2.1 We say that a subset Γ of $\mathbf{R}^n$ is a *separating hypersurface* if Γ is an embedded, oriented, connected, $(n-1)$-dimensional sub-manifold and $\mathbf{R}^n \setminus \Gamma$ has two connected components. If Γ is a separating hypersurface let C^i, $i = 1, 2$ be the two connected components of $\mathbf{R}^n \setminus \Gamma$ and let $\mathbf{n}(x)$ be a unit normal vector on Γ which defines the orientation such that C^1 is on the "positive side". That is, if $p \in \Gamma$ and we choose local coordinates (W, φ) centered at p in which $W \cap \Gamma$ is the set $\{(x_1, \ldots, x_n) \in W : x_1 = \ldots = x_{n-1} = 0\}$ and $\mathbf{n}(x) = [0, \ldots, 0, 1]^T$ then $C^1 \cap W = \{(x_1, \ldots, x_n) \in W : x_n > 0\}$. We say that a vector field f *points towards* C^1 *on* Γ if $\forall x \in \Gamma\ \mathbf{n}^T(x) f(x) > 0$.

Definition 2.2 Let N be an open subset of $\mathbf{R}^n$ with $0 \in \overline{N}$. A vector field f on N is *asymptotically stable on* N if for each neighborhood $\mathcal{V}$ of 0 there is a neighborhood $\mathcal{W}$ of 0 such that $\forall x \in \mathcal{W} \cap N$ the integral curve of f starting at x, $\phi(x, t)$, is defined for all $t > 0$, $\phi(x, t) \in \mathcal{V} \cap N$, and $\lim_{t \to \infty} \phi(x, t) = 0$. We say that the vector field f is *globally asymptotically stable (relative to N)* if it is asymptotically stable and for all $x \in N$ $\lim_{t \to \infty} \phi(x, t) = 0$.

3. Main Theorem

Definition 3.3 Let M be an open connected subset of $\mathbf{R}^n$. Given the system

$$(\Sigma) \qquad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

a function $V : \overline{M} \longrightarrow \mathbf{R}$ is called a *control Lyapunov function (clf)* for Σ on M if the following hold:

(i) V is continuous on $\overline{M}$, and smooth (i.e. infinitely differentiable) on M

(ii) V is positive definite and proper on $\overline{M}$

(iii) $\inf_{u \in \mathbf{R}^m} \{L_f V(x) + u_1 L_{g_1} V(x) + \cdots + u_m L_{g_m} V(x)\} < 0$ for each $x \in M$.

Remark 3.1 Notice that since M could be a proper subset of $\mathbf{R}^n$ a trajectory of Σ could "escape" from M while satisfying the Lyapunov property (iii). The proof of the main theorem below requires a careful study of precisely under which conditions these "escapes" occur.

The following theorem gives an explicit formula for the construction of a piecewise continuous stabilizing feedback law.

Theorem 3.1 *Let* M^j, $j = 1, 2$, *be connected open subsets of* $\mathbf{R}^n$ *such that* $M^1 \cup M^2 = \mathbf{R}^n \setminus \{0\}$. *Consider the system* (Σ) *as above, with* f, g_i *smooth, and* $f(0) = 0$. *Suppose there exists a separating hypersurface* Γ *with* $0 \in \Gamma$, $\Gamma \setminus \{0\} \subset M^1 \cap M^2$. *Let* C^1, C^2 *be the two connected components of* $\mathbf{R}^n \setminus \Gamma$, *with* $C^j \subset M^j$, *for* $j = 1, 2$. *Let* $V^j : M^j \longrightarrow \mathbf{R}^n$ *be control Lyapunov functions for (Σ) on* M^j.

Assume the following "transversality conditions" hold:

(T1) $f(x)$ *is tangent to* Γ *for all* $x \in \Gamma$,

(T2) For each $x \in \Gamma \setminus \{0\}$ *(at least) one of the following holds:*

1. *The vector* $-L_{g_i} V^1(x) \cdot g_i(x)$ *points to* C^1 *for* $i = 1, \ldots, m$. *We let* Γ_1 *be the set of such points* x.
2. *The vector* $-L_{g_i} V^2(x) \cdot g_i(x)$ *points to* C^2 *for* $i = 1, \ldots, m$. *We let* Γ_2 *be the set of such points* x.

Finally assume

(E) if $x \in \Gamma_1$ *then* $V^1(x) \le V^2(x)$ *and if* $x \in \Gamma_2$ *then* $V^2(x) \le V^1(x)$.

Then there exists a globally asymptotically stabilizing feedback law which is smooth on $C^1 \cup C^2$.

Before proving the theorem we make several remarks.

Remark 3.2 Conditions (T1) and (T2) will guarantee (because of the form of the feedback law) that trajectories do not remain on Γ. They will not, however, guarantee that trajectories remain on either C^1 or C^2. In fact an infinite number of switchings between those two regions may occur.

Remark 3.3 Condition (E) guarantees that when a trajectory changes regions the "energy" as measured by the new Lyapunov function does not increase. This condition is rather intuitive and is similar to others considered in the literature [4]. However it differs significantly in that it does not require knowledge of the solution to the differential equation. It should be emphasized that what makes this condition useful is the existence of an explicit formula for the stabilizing feedback.

Remark 3.4 Condition (E) could be relaxed somewhat since if $x \in \Gamma_1 \cap \Gamma_2$, then the state x can't be reached from either C^1 or C^2 with trajectories from the closed loop system.

Remark 3.5 If f, g_i, V^j are real analytic then the resulting feedback law is real analytic on $C^1 \cup C^2$.

PROOF. (Sketch) The proof will follow the lines of the proof in [6]. However, here we will need to pay special attention to trajectories which switch between the regions C^1 and C^2.

Consider the set $S = \{(a, b) \in \mathbf{R}^2 : b > 0 \text{ or } a < 0\}$. Define

$$\psi(a, b) = \begin{cases} \frac{a + \sqrt{a^2 + b^2}}{b} & b \neq 0 \\ 0 & b = 0 \end{cases}$$

Then ψ is analytic on S. (The function $p = \psi(a, b)$ is a solution of $bp^2 - 2ap - b = 0$ for $(a, b) \in S$.) Define for $j = 1, 2$, $i = 1, \ldots, m$, $x \in M^j$, the functions

$$\alpha^j(x) = L_f V^j(x)$$

$$b_i^j(x) = L_{g_i} V^j(x)$$

$$\beta^j(x) = \sum_{i=1}^{m} (b_i^j(x))^2$$

The third condition in the definition of clf, applied to V^j, M^j is equivalent to asking that $\beta^j(x) = 0 \implies \alpha^j(x) < 0$. That is, $(\alpha^j(x), \beta^j(x)) \in S$ for $j = 1, 2$.

We define feedback laws $k^j = (k_1^j, \ldots, k_m^j)$, $j = 1, 2$ by

$$k_i^j(x) = \begin{cases} -b_i^j(x)\psi(\alpha^j(x), \beta^j(x)) & x \neq 0 \;\; x \in M^j \\ 0 & x = 0 \end{cases}$$

Notice that $0 \in \overline{M^j}$. The function k^j is smooth on M^j for $j = 1, 2$. If f, g_i as well as V^j are real analytic then k^j is real analytic. On each M^j this is the same universal formula used in the smooth case (see [7]).

We now define vector fields h^j on M^j, for $j = 1, 2$ by $h^j(x) = f(x) + \sum_{i=1}^{m} k_i^j(x) g_i(x)$. These vector fields are smooth on M^j. By the transversality conditions (T1) (T2), $h^j(x)$ points into C^j on Γ_j for $j = 1, 2$.

Notice also that the function V^j is a Lyapunov function for h^j on C^j, $j = 1, 2$. Indeed

$$\begin{aligned} & L_{h^j} V(x) & (1) \\ &= L_f V^j(x) + k_1^j L_{g_1} V(x) + \cdots + k_m^j L_{g_m} V(x) \\ &= \alpha^j(x) - \beta^j(x)\psi(\alpha^j(x), \beta^j(x)) \\ &= -\sqrt{(\alpha^j(x))^2 + (\beta^j(x))^2} \quad < \quad 0 \end{aligned}$$

The main technical difficulty is due to the fact that a trajectory of h^j might leave C^j. But if it does, it must do so through Γ.

We define the feedback law $k(x)$ as follows

$$k(x) = \begin{cases} k^j(x) & x \in C^j \quad j = 1, 2 \\ k^1(x) & x \in \Gamma_1 \\ k^2(x) & x \in \Gamma_2 \setminus \Gamma_1 \end{cases}$$

Define h to be the closed loop vector field. That is $h(x) = f(x) + \sum_{i=1}^{m} k_i(x) g_i(x)$.

To prove stability let $r > 0$ be given. Let ρ^j be the minimum value of $V^j(x)$ on $\partial B(0, r) \cap M^j$, for $j = 1, 2$ and set $\rho = \min(\rho^1, \rho^2)$ (where $B(0, r)$ denotes the open ball of radius r centered at 0). Notice that $\rho > 0$ because V^j is positive definite for $j = 1, 2$. Take $\gamma \in (0, \rho)$ and define

$$K_r = \{x \in B(0, r) \cap (C^1 \cup \Gamma) : V^1(x) \leq \gamma\} \quad \cup$$
$$\{x \in B(0, r) \cap (C^2 \cup \Gamma) : V^2(x) \leq \gamma\}$$

First we prove that K_r is closed. Let $x_n \in K_r$ for all n, and $\hat{x} = \lim x_n$. There are three possibilities: (a) $\hat{x} \in C^1$, (b) $\hat{x} \in C^2$, or (c) $\hat{x} \in \Gamma$. Cases (a) and (b) are treated in the same way. So, suppose $\hat{x} \in C^1$. Then for some N_1, we have $x_n \in C^1$ for $n > N_1$. And hence $V^1(x_n) \leq \gamma < \rho$. By continuity of V^1, we get $V^1(\hat{x}) \leq \gamma < \rho$. Therefore, $\hat{x} \in C^1 \cap B(0, r)$, and hence $\hat{x} \in K_r$. If $\hat{x} \in \Gamma$ there must exist a subsequence of $\{x_n\}$ completely contained in either C^1 or C^2. Assume $\{x_{n_k}\} \subset C^1$ is such a subsequence. Then $V^1(x_{n_k}) \leq \gamma$ and by continuity $V^1(\hat{x}) \leq \gamma$. So again $\hat{x} \in K_r$.

Next we prove that K_r is invariant, that is, a trajectory starting in K_r remains in K_r. Let $\eta(t)$ be such a trajectory and suppose that for some $t > 0$, $\eta(t) \notin K_r$. Let $t_0 = \sup\{t : \eta(s) \in K_r \text{ for } 0 \leq s < t\}$. There are three cases: (a) $\eta(t_0) \in C^1$, (b) $\eta(t_0) \in C^2$, and (a) $\eta(t_0) \in \Gamma$. Assume that $\eta(t_0) \in C^1$ (the case $\eta(t_0) \in C^2$ is treated analogously). Then for t near t_0, $\eta(t) \in C^1$. For x near $\eta(t_0)$ the vector field h coincides with h^1. By the Lyapunov property (iii) $V^1(\eta(t))$ is decreasing for t near t_0 and therefore for some $\varepsilon > 0$ and $0 < t - t_0 < \varepsilon$ we get $\eta(t) \in B(0, r)$ and $V^1(\eta(t)) < V^1(\eta(t_0)) \leq \gamma$. But then $\eta(t) \in K_r$ for $t_0 < t < t_0 + \varepsilon$, which is a contradiction.

Assume then that $\eta(t_0) \in \Gamma$. Since K_r is closed, $\eta(t_0) \in K_r$. Then, either $V^1(\eta(t_0)) \leq \gamma$, or $V^2(\eta(t_0)) \leq \gamma$. From condition (E) we conclude that $V^j(\eta(t_0)) \leq \gamma$ for $\eta(t_0) \in \Gamma_j$, $j = 1, 2$. If $\eta(t_0) \in \Gamma_1$ then $h(\eta(t_0))$ points toward C^1. So, for small $\varepsilon > 0$, $\eta(t) \in C^1$ for $0 < t - t_0 < \varepsilon$. By the Lyapunov property (1) we get $V^1(\eta(t)) \leq \gamma$ for all such t. This implies that for such t $\eta(t) \in K_r$ which contradicts the definition of t_0. An analogous argument applies if $\eta(t_0) \in \Gamma_2$. This concludes the proof that K_r is invariant.

By continuity of V^j, we can find $\delta > 0$ such that if $\|x\| < \delta$ and $x \in M^j$ then $V^j(x) \leq \gamma$. This proves stability.

Since K_r is also bounded, it is therefore compact. Being also invariant we conclude that trajectories starting in K_r are defined for all $t > 0$.

To prove asymptotic stability we must show that trajectories tend to 0 as t tends to infinity.

Let $\eta(t)$ be a trajectory of the closed loop system with $\eta(0) \in C^1$. Let $\tau_0 = \sup\{t : \eta(s) \in C^1$ for $0 < s < t\}$. By properness of V^1 the trajectory remains in $\{x : V^1(x) \leq V^1(\eta(0))\}$ and is defined for all t as long as it stays in C^1. Therefore, either $\tau_0 = \infty$ and so $\lim_{t\to\infty} \eta(t) = 0$ (by the standard arguments) or $\eta(\tau_0) \in \Gamma$. Moreover, $\eta(\tau_0) \notin \Gamma_1$ since $h^1(\eta(\tau_0))$ must point towards C^2 in order for η to reach Γ. Therefore, by the transversality conditions, we must have $\eta(\tau_0) \in \Gamma_2$. This guarantees that the trajectory can be continued beyond τ_0 since $h(\eta(\tau_0)) = h^2(\eta(\tau_0))$, and $h^2(\eta(\tau_0))$ points into C^2. Repeating the process either $\eta(t)$ is in C^2 for all t and then converges to 0, or we can find τ_1 such that $\eta(\tau_1) \in \Gamma_1$ and $\eta(t) \in C^2$ for $\tau_0 < t < \tau_1$. Continuing in this way, either the trajectory stays in C^j for some j or we obtain a sequence $\tau_0, \tau_1, \ldots, \tau_n, \ldots$, such that $\eta(\tau_n) \in \Gamma$. By the Lyapunov property (E) it is easy to show that $V^j(\eta(\tau_{i+1})) < V^j(\eta(\tau_i))$ for $j = 1, 2$ and all i.

To complete the proof we must show that $\eta(\tau_n) \longrightarrow 0$. We prove this by showing that $V^j(\eta(\tau_n)) \longrightarrow 0$. This is done following standard arguments. If any subsequence $V^j(\eta(\tau_{n_k}))$ stabilizes at $\gamma_1 \neq 0$ then ∇V^j must be zero at some point on $\Gamma \setminus \{0\}$ which violates the transversality conditions. ∎

4. Bounded Controls

Using the formula presented in [11] we can guarantee certain bounds on the controls.

Let $\mathcal{B}_m = \{u : \|u\|^2 = u_1^2 + \cdots + u_m^2 < 1\}$.

Definition 4.4 Let M and Σ be as in Definition 3.3. A function $V : \overline{M} \longrightarrow \mathbf{R}$ is called a *control Lyapunov function with controls in $\mathcal{B}_m$ (clfb)* for Σ on M if the conditions of Definition 3.3 hold with (iii) replaced by

(iii') $\inf_{u \in \mathbf{R}^m}\{L_f V(x) + u_1 L_{g_1} V(x) + \cdots + u_m L_{g_m} V(x)\} < 0$ for each $x \in M$ and $u \in \mathcal{B}_m$.

Theorem 4.2 *With the same hypothesis as in Theorem 3.1 but replacing clf with clfb we obtain a globally asymptotically stabilizing feedback law $u = k(x)$ which satisfies $k(x) \in \mathcal{B}_m$.*

PROOF. The proof follows the same pattern as that of Theorem 3.1 where instead of using the function ψ we use the function ξ defined by

$$\xi(a, b) = \begin{cases} \frac{a + \sqrt{a^2 + b^2}}{b(1 + \sqrt{1 + b})} & b > 0 \\ 0 & b = 0 \end{cases}$$

The proof that this function achieves the desired bounds is given in [11]. ∎

5. Conclusions

We proved a theorem which guarantees global asymptotic stability under more general conditions than previously reported. The conditions do not require knowledge of the solution to the differential equation (or the switching times between regions). The construction of the feedback law is quite explicit from the Lyapunov functions given.

A second theorem gives bounded feedback controls provided that the Lyapunov property can be veryfied using controls in the unit ball.

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