< HW #1 >

< 4.5 Exercises >

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$$A = A_0\left(1+\frac{r}{n}\right)^{nt},$$ where A_0 initial amount of money invested,

r : interest rate

compounded n times a year.

t : # years

$\lim_{n\to\infty} A$

$$= \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}.$$

Let $y = \left(1+\frac{r}{n}\right)^{nt}$. Then $\ln y = nt \ln\left(1+\frac{r}{n}\right)$.

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1+\frac{r}{n}\right)$, This has $\infty \cdot 0$ form.

$$= t\lim_{n\to\infty} \frac{\ln\left(1+\frac{r}{n}\right)}{1/n} = t\cdot\lim_{n\to\infty}\frac{\frac{\cancel{-}r/\cancel{n^2}}{1+r/n}}{\cancel{-}\frac{1}{\cancel{n^2}}} = t\lim_{n\to\infty}\frac{r}{1+r/n} = tr$$

Then $\lim_{n\to\infty} y = e^{rt}$ and so $\lim_{n\to\infty} A = A_0 e^{rt}$.

<58> $V = \frac{mg}{c}\left(1 - e^{-ct/m}\right)$, where g acceleration due to gravity, c a proportionality const, m mass, t seconds.
(air resistance)

(a) $\lim_{t \to \infty} V = \frac{mg}{c} \lim_{t \to \infty} \left(1 - e^{-ct/m}\right) = \frac{mg}{c}$, the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim_{c \to 0^+} v = \lim_{c \to 0^+} \frac{mg}{c}\left(1 - e^{-ct/m}\right)$

$$= mg \lim_{c \to 0^+} \frac{1 - e^{-ct/m}}{c} \quad \left(\text{This has } \tfrac{0}{0} \text{ type.}\right)$$

$$= mg \lim_{c \to 0^+} \frac{-e^{-ct/m} \cdot (-t/m)}{1}$$

$$= mg \lim_{c \to 0^+} \frac{t}{m} \cdot e^{-ct/m} = gt.$$

That is, the velocity of a falling object in a vaccum is directly proportional to the amount of time it falls.

<60> $v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$, where r is the radius of a metal cable
R distance from the center of the cable to the exterior of the insulation
c a positive const.

(a) $\lim_{R \to r^+} v = \lim_{R \to r^+} -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right) = -c \lim_{R \to r^+} \left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right) = -c \cdot 1 \cdot 0 = 0.$

As the insulation of a metal cable becomes thinner, the velocity of an electrical impulse in the cable approaches zero.

(b) $\lim_{r \to 0^+} v = -c \lim_{r \to 0^+} \left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$ has $0 \cdot (-\infty)$ type.

$$= -\frac{c}{R^2} \lim_{r \to 0^+} \frac{\ln(r/R)}{1/r^2} = -\frac{c}{R^2} \lim_{r \to 0^+} \frac{\frac{1/R}{r/R}}{\frac{-2}{r^3}} = -\frac{c}{R^2} \lim_{r \to 0^+} \frac{-r^2}{2} = 0.$$

As the radius of the metal cable approaches zero, the velocity of an electrical impulse in the cable approaches zero.

<61> $\lim\limits_{x \to a} \dfrac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$, where $a > 0$.

(has $\frac{0}{0}$ type).

$$= \lim_{x \to a} \frac{\frac{1}{2}(2a^3x - x^4)^{-\frac{1}{2}}(2a^3 - 4x^3) - a\left(\frac{1}{3}\right)(a^2x)^{-2/3} \cdot a^2}{\cancel{1} - \frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{\frac{1}{2}(2a^4 - a^4)^{-\frac{1}{2}}(2a^3 - 4a^3) - \frac{1}{3}a(a^3)^{-2/3} \cdot a^2}{\cancel{1} - \frac{1}{4}(a^4)^{-3/4}(3a^3)}$$

$$= \frac{-a^3 \cdot a^{-2} - \frac{1}{3}a\,\cancel{a^{-2}} \cdot \cancel{a^2}}{\cancel{1} - \frac{3}{4}} = \frac{-a - \frac{1}{3}a}{-\frac{3}{4}}$$

$$= \frac{-4a - \frac{4}{3}a}{-3}$$

$$= \cancel{\frac{(12+4)}{3}a}$$

$$= \frac{\frac{16a}{3}}{3} = \frac{16}{9}a.$$

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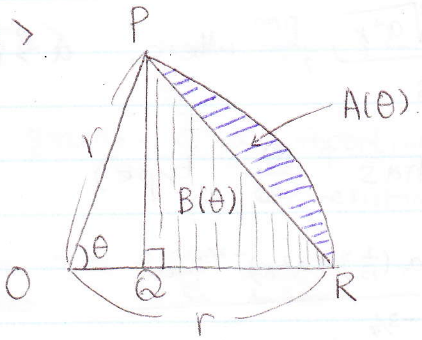

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$

$A(\theta)$ = area of the sector of the circle − Area of ΔOPR.

$$= \frac{1}{2} r^2 \theta - \frac{1}{2} r \cdot |PQ| = \frac{1}{2} r^2 \theta - \frac{1}{2} r \cdot r \sin\theta$$

$$= \frac{r^2}{2} (\theta - \sin\theta).$$

$$B(\theta) = \frac{1}{2} |QR|\,|PQ|$$

$$= \frac{1}{2} (r - r\cos\theta)\, r\sin\theta = \frac{1}{2} r^2 (1-\cos\theta)\sin\theta.$$

$$\lim_{\theta \to 0+} \frac{A(\theta)}{B(\theta)} = \lim_{\theta \to 0+} \frac{\frac{r^2}{2}(\theta - \sin\theta)}{\frac{1}{2} r^2 (1-\cos\theta)\sin\theta} = \lim_{\theta \to 0+} \frac{(\theta - \sin\theta)}{(1-\cos\theta)\sin\theta}$$

(has $\frac{0}{0}$ type.)

$$= \lim_{\theta \to 0+} \frac{1-\cos\theta}{\sin\theta \cdot \sin\theta + (1-\cos\theta)\cos\theta} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{\theta \to 0+} \frac{\sin\theta}{2\sin\theta\cos\theta + \sin\theta\cos\theta + (1-\cos\theta)(-\sin\theta)}$$

$$= \lim_{\theta \to 0+} \frac{\sin\theta}{4\sin\theta\cos\theta - \sin\theta} = \lim_{\theta \to 0+} \frac{1}{4\cos\theta - 1} = \frac{1}{3}.$$