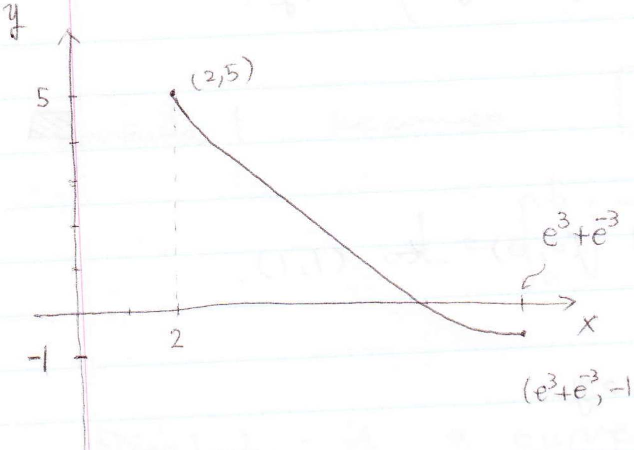
HW#9 : Section 6.3

<6> $x = e^{t} + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$.



$$\frac{dx}{dt} = e^{t} - e^{-t}$$

$$\frac{dy}{dt} = -2.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} - 2 + e^{-2t} + 4$$

$$= e^{2t} + 2 + e^{-2t}$$

$$= (e^{t} + e^{-t})^2$$

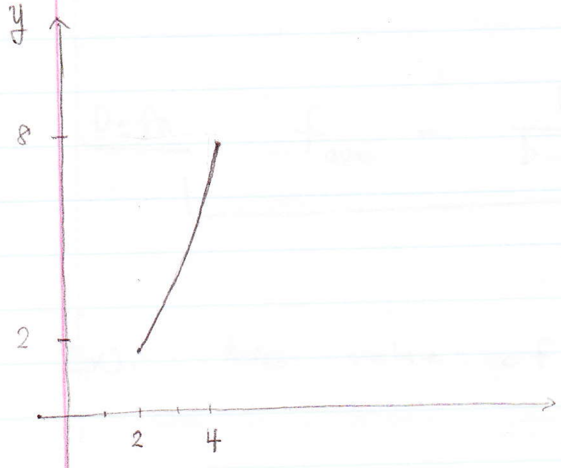
and $L = \int_0^3 \sqrt{(e^{t}+e^{-t})^2}\, dt$

$$= \int_0^3 (e^{t}+e^{-t})\, dt$$

$$= [e^{t} - e^{-t}]_0^3 = e^3 - e^{-3} - (1-1)$$

$$= e^3 - e^{-3}.$$

$\langle 8 \rangle \quad y = \frac{x^2}{2} - \frac{\ln x}{4}, \quad 2 \le x \le 4.$



$$\frac{dy}{dx} = x - \frac{1}{4x}$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(x - \frac{1}{4x}\right)^2$$
$$= 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}$$
$$= x^2 + \frac{1}{2} + \frac{1}{16x^2}$$
$$= \left(x + \frac{1}{4x}\right)^2$$

So, $$L = \int_2^4 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx$$

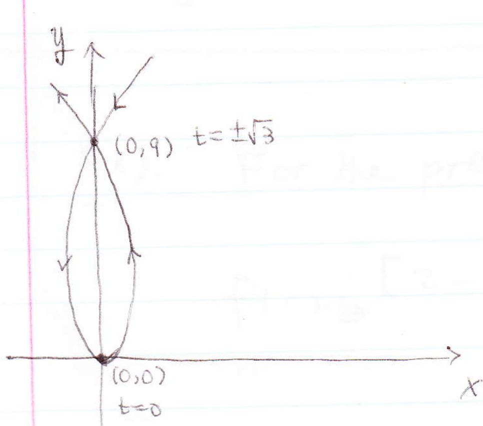
$$= \int_2^4 \left(x + \frac{1}{4x}\right) dx$$

$$= \left[\frac{1}{2}x^2 + \frac{\ln x}{4}\right]_2^4$$

$$= 8 + \frac{2\ln 2}{4} - \left(2 + \frac{\ln 2}{4}\right)$$

$$= 6 + \frac{\ln 2}{4}. \quad \blacksquare$$

$\langle 14 \rangle \quad x = 3t - t^3, \quad y = 3t^2$



$$\frac{dx}{dt} = 3 - 3t^2; \quad \frac{dy}{dt} = 6t.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3-3t^2)^2 + (6t)^2 = (3+3t^2)^2$$

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3+3t^2)\, dt = 2\int_0^{\sqrt{3}} (3+3t^2)\, dt$$

$$= 2\left[3t + t^3\right]_0^{\sqrt{3}}$$

$$= 2\left(3\sqrt{3} + 3\sqrt{3}\right) = 12\sqrt{3}. \quad \blacksquare$$

< HW#8 : Section 6.4 >

<3> $f(t) = t e^{t/2}$, $[0, 3]$.

Ans) $f_{ave} = \frac{1}{3-0} \int_0^3 t e^{t/2} \, dt$. Using the integration by parts,

$\left[\begin{array}{ll} u = t, & dv = e^{t/2} dt \\ du = dt, & v = 2e^{t/2} \end{array} \right]$

$$\int t e^{t/2} \, dt = 2t e^{t/2} - \int 2 e^{t/2} \, dt$$

$$\Rightarrow f_{ave} = \frac{1}{3} \left\{ \left[2t e^{t/2} \right]_0^3 - \left[4 e^{t/2} \right]_0^3 \right\}$$

$$= \frac{1}{3} \left\{ 6e^{3/2} - 4e^{3/2} + 4 \right\}$$

$$= \frac{1}{3} \left(2e^{3/2} + 4 \right).$$

<8> $f(x) = \frac{2x}{(1+x^2)^2}$, $[0, 2]$.

Ans)

a) $f_{ave} = \frac{1}{2-0} \int_0^2 \frac{2x}{(1+x^2)^2} \, dx$

$$= \frac{1}{2} \int_1^5 \frac{1}{u^2} \, du \qquad [u = 1+x^2, \; du = 2x\,dx]$$

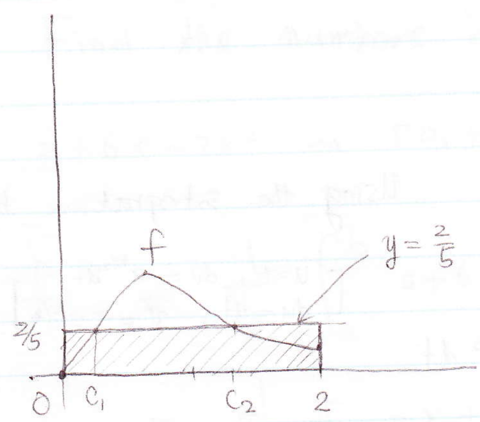
$$= \frac{1}{2} \left[-\frac{1}{u} \right]_1^5 = -\frac{1}{2} \left(\frac{1}{5} - 1 \right) = \frac{2}{5}.$$

(b) $f(c) = f_{ave} \Leftrightarrow \frac{2c}{(1+c^2)^2} = \frac{2}{5} \Leftrightarrow 5c = (1+c^2)^2$

$\Leftrightarrow \quad c_1 \approx .220$ or $c_2 \approx 1.207$

<8> continued

c)



<14>

$$T_{ave} = \frac{1}{30-0}\int_0^{30} 20+75e^{-t/50}\,dt$$

$$= \frac{1}{30}\left[20t - 50\cdot 75e^{-t/50}\right]_0^{30}$$

$$= \frac{1}{30}\left[\left(600 - 3750e^{-3/5}\right) - (-3750)\right]$$

$$= \frac{1}{30}\left(4350 - 3750e^{-3/5}\right) = 145 - 125e^{-3/5} \approx 76.4^\circ C$$

< HW #9: Section 6.7 >

<4> (a) For $0 \le x \le 1$, $f(x) = kx^2(1-x) \ge 0$ iff $k \ge 0$. (if and only if)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 kx^2(1-x)\,dx = k\int_0^1 (x^2 - x^3)\,dx$$

$$= k\left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 = k/12.$$

$$k/12 = 1 \iff k = 12.$$

Therefore, f is a prob. density fn. if and only if $k=12$.

(b) $k = 12$.

$$P\left(X \ge \tfrac{1}{2}\right) = \int_{\frac{1}{2}}^{\infty} f(x)\,dx = \int_{\frac{1}{2}}^{1} 12x^2(1-x)\,dx$$

$$= \int_{\frac{1}{2}}^{1} 12x^2 - 12x^3\,dx$$

$$= \left[4x^3 - 3x^4\right]_{\frac{1}{2}}^{1}$$

$$= (4-3) - \left(\tfrac{1}{2} - \tfrac{3}{16}\right) = 1 - \tfrac{5}{16} = \tfrac{11}{16}.$$

(c) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_0^1 x \cdot 12x^2(1-x)\,dx = 12\int_0^1 x^3 - x^4\,dx$$

$$= 12\left[\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^1$$

$$= 12\left(\tfrac{1}{4} - \tfrac{1}{5}\right) = \tfrac{3}{5}.$$

$\square$

<8> a) $\mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & , t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0. \end{cases}$

i) $P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000} e^{-t/1000}\, dt = \left[-e^{-t/1000} \right]_0^{200}$

$= -e^{-1/5} + 1 \approx .181$

ii) $P(X > 800) = \int_{800}^{\infty} \frac{1}{1000} e^{-t/1000}\, dt$

$= \lim_{x \to \infty} \left[-e^{-t/1000} \right]_{800}^{x} = 0 + e^{-4/5} \approx .449$

b) $\int_m^{\infty} f(t)\, dt = \frac{1}{2} \Rightarrow \lim_{x \to \infty} \int_m^x \frac{1}{1000} e^{-t/1000}\, dt = \frac{1}{2}$

$\Rightarrow \lim_{x \to \infty} \left[-e^{-t/1000} \right]_m^x = \frac{1}{2}$

$\Rightarrow 0 + e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln \frac{1}{2}$

$\Rightarrow m = -1000 \ln \frac{1}{2} = 1000 \ln 2$

≈ 693.1

<10> a) $\mu = 69$, $\sigma = 2.8$, $P(65 \le X \le 73)$

$$= P\left(\frac{65-69}{2.8} \le \frac{X-69}{2.8} \le \frac{73-69}{2.8}\right)$$

$$= P(-1.429 \le Z \le 1.429)$$

Using the standard normal table, $\approx 2 * (.9236 - .5) = 2 * (.4236) = .8472$.

b) $P(X > 6 \text{ feet}) = P(X > 72 \text{ inches})$

$$= 1 - P(-\infty \le X \le 72)$$

$$= 1 - P\left(-\infty \le \frac{X-69}{2.8} \le \frac{72-69}{2.8}\right)$$

$$= 1 - P(-\infty \le Z \le 1.071)$$

$\cong 1 - .8577 = .1423$. about 14%.

<12> a) $P(-\infty \le X \le 480)$

$$= P\left(-\infty \le \frac{X-500}{12} \le \frac{480-500}{12}\right)$$

$= P(-\infty \le Z \le -1.67) = .0475 \approx 4.8\%$.

b) We need to find μ so that $P(X < 500) = .05$.

$$P(X < 500) = P\left(\frac{X-\mu}{12} \le \frac{500-\mu}{12}\right) = .05$$

$$= P\left(Z \le \frac{500-\mu}{12}\right)$$

$$\Rightarrow \frac{500-\mu}{12} = -1.645$$

$\Rightarrow \mu = 500 + 12(1.645) = \cancel{480.26}$ 519.74.

So the target weight is at least 519.74g.