Example: Back to urn problem.

4B
3R
3Y

Draw three chips in succession, at random and without replacement. What is prob. we ~~seze~~ obtain the sequence

$$B_1 R_2 R_3$$

Solution (a) $\dfrac{4 \times 3 \times 2}{10 \times 9 \times 8} = \dfrac{24}{720}$

Solution (b) Use the multiplication rule for the $P(A \cap B)$ that results from the definition of conditional probability:

$$P(B_1 \text{ and } R_2 \text{ and } R_3) = P(B_1) \times P(R_2 \text{ and } R_3 \mid B_1)$$

$$= \boxed{P(B_1) \times P(R_2 \mid B_1) \times P(R_3 \mid B_1 \text{ and } R_2)}$$

$$= \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8}$$

Notion of Independence:

Two events A and B are said to be independent when and only when the occurrence of one does NOT effect the probability of the other occurring. I.e.

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B)$$

In the Urn example, drawing without replacement creates dependent events.

But drawing with replacement yields independent events, since the probabilities are ~~constant~~ unchanged from draw to draw.

with replacement

$P(B_1 \text{ and } R_2 \text{ and } R_3)$

$= P(B_1) \times P(R_2|B_1) \times P(R_3|B_1 R_2)$

$= \frac{4}{10} \times \frac{3}{10} \times \frac{3}{10}$.

Equivalent definition.

Two events A & B are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B).$$

Note:

P(A) often called "prior" probability

P(A|B) often called the "updated" or "posterior" probability. Updated after we learn event B has occurred.

In Johns Landing Housing plan, 42% of the houses have a deck and a garage; 60% have a deck. Find the probability that a home chosen at random from this housing area has a garage, given it has a deck.

	D	D′
G	.42	
G′	.18	
	.60	.40

$$P(G|D) = \frac{P(G \text{ and } D)}{P(D)} = \frac{.42}{.60} = \frac{7}{10} = .7 = 70\%.$$

In a large shopping mall, a marketing agency conducted a survey on credit cards. The Results are shown in the following two-way contingency table.

credit card status

EMPLOYMENT Status		Has	DOES NOT HAVE		
	Employed	18	29	47	← marginal events
	Unemployed	28	34	62	←
		46	63	109	
		↑	↑		

marginal events.

The four cells are called joint events.

Let E = employed $\qquad$ H = has a credit card

E' = not employed $\qquad$ H' = no credit card

$$P(E \text{ and } H) = \frac{18}{109} = \text{joint prob. / cell prob.}$$

$$P(E) = \frac{47}{109} = \text{marginal prob.}$$

$$P(H|E) = \frac{P(H \text{ and } E)}{P(E)} = \frac{\frac{18}{109}}{\frac{47}{109}} = \frac{18}{47}$$

	H	H'	
E	$\frac{18}{109}$	$\frac{29}{109}$	$\frac{47}{109}$
E'	$\frac{28}{109}$	$\frac{34}{109}$	$\frac{62}{109}$
	$\frac{46}{109}$	$\frac{63}{109}$	

The four cell prob. sum to 1.

Both sets of marginal prob add to 1.

We have three probability distributions here.

Two marginal:

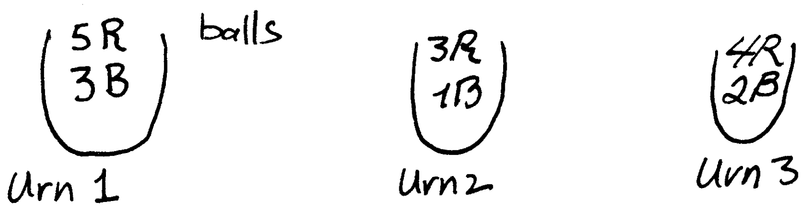
E	E'
$\frac{47}{109}$	$\frac{62}{109}$

H	H'
$\frac{46}{109}$	$\frac{63}{109}$

One joint distribution

EH	EH'	$E'H$	$E'H'$
$\frac{18}{109}$	$\frac{29}{109}$	$\frac{28}{109}$	$\frac{63}{109}$

Here we use relative frequency to provide us with the probability assignments.

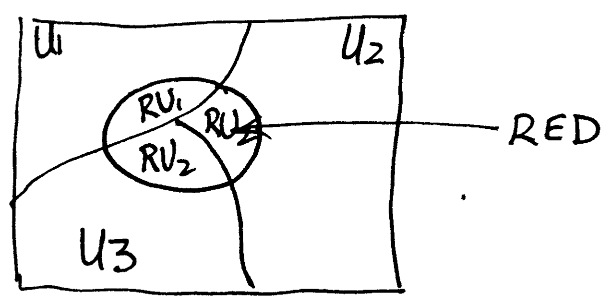


Experiment: If an urn is selected at random an a ball is drawn, find the probability it will be red.

classify a ball by URN and COLOR.

		COLOR		
		RED	BLACK	
URN	URN1	5	3	8
	URN2	3	1	4
	URN3	4	2	6
		12	6	18

Venn Diagram



Note: RU_1, RU_2 and RU_3 are all mutually exclusive and unioned together give Red.

RED = (R and U_1) or (R and U_2) or (R and U_3)

apply addition Rule → $P(RED) = P(R \text{ and } U_1) + P(R \text{ and } U_2) + P(R \text{ and } U_3)$

Now apply multiplication Rule

$= P(R|U_1)P(U_1) + P(R|U_2)P(U_2) + P(R|U_3)P(U_3)$

$= \frac{5}{8}\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{1}{3} + \frac{4}{6}\cdot\frac{1}{3}$

$= 49/72 \neq 48/72$. (but close).

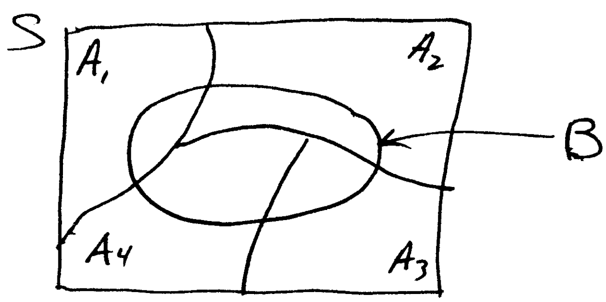
Bayes's Theorem

~~Examples follow on orange handout sheet~~

Consider the four mutually exclusive events A_1, A_2, A_3, and A_4 that partition S; that is

$$A_i \cap A_j = \emptyset \quad i \neq j, \quad \text{and} \quad S = A_1 \cup A_2 \cup A_3 \cup A_4 .$$

The Venn Diagram looks like



Let B be any event. A very general picture of this is to the left. It overlaps all four A_i's.

As before, $B = BA_1$ or BA_2 or BA_3 or BA_4

So, $P(B) = P(BA_1) + P(BA_2) + P(BA_3) + P(BA_4)$

$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) + P(B|A_4)P(A_4)$

$$= \sum_{i=1}^{4} P(B|A_i)\cdot P(A_i).$$

Now, for example:

$$P(A_2|B) = \frac{P(B \text{ and } A_2)}{P(B)}$$

$$P(A_2|B) = \frac{P(B|A_2)\cdot P(A_2)}{\sum_{i=1}^{4} P(B|A_i)\cdot P(A_i)}$$

BAYES' RULE

$P(A_i)$ is called a prior prob.

$P(A_i|B)$ is called an updated or posterior prob.

Bayes's rules gives us a method of computing updated prob. (revised prob.) when new information is learned.

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Applications of Bayes' Theorem

1. Suppose that you are walking down the street and notice that the Dept. of Public Health is giving a free medical test for a certain disease. The test is 90% reliable in the following sense: If a person has the disease, there is a probability of 0.9 that the test will give a positive response; whereas, if the person does not have the disease, there is a probability of 0.1 that the test will give a positive response.

   Data indicate that your chances of having the disease are only 1 in 10,000; that is, 0.0001. However, since the test costs you nothing and is fast and harmless, you decide to take the test. A few days later you learn that you had a positive response to the test. What is now the probability that you have the disease?

2. Three different machines M_1, M_2, M_3 were used for producing a large batch of similar manufactured items. Suppose 20% of the items are produced by M_1, 30% of the items by M_2, and 50% by M_3. Suppose further that 1% of the items produced by M_1 are defective, 2% of the items produced by M_2 are defective, and 3% of the items produced by M_3 are defective. Finally, suppose that one item is selected at random from the entire batch and it is found to be defective. Determine the probability that the item was produced by M_2.

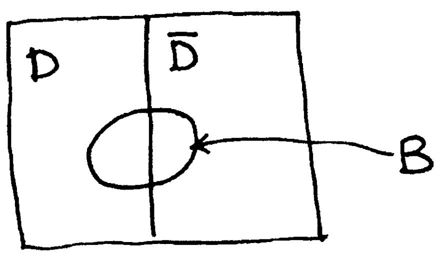
Picture for example 1.

Let D = event you have the disease,

$\bar{D}$ = event you do not have the disease.

B = event you tested positive.

We seek, $P(D|B)$.



D and $\bar{D}$ partition sample space S as $D \cap \bar{D} = \emptyset$ and $D \cup \bar{D} = S$.

Bayes' Theorem:

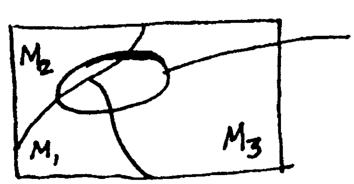
$$P(D|B) = \frac{P(B|D)P(D)}{P(B|D)P(D) + P(B|\bar{D})P(\bar{D})}.$$

Now, given $P(D) = .0001$, $P(B|D) = .90$ (reliability),

$P(B|\bar{D}) = 0.10$. $P(\bar{D}) = .9999$

Now, plug these numbers into the above expression from Bayes' Theorem.

Hint for example 2.



B = event the item is defective.

Find: $P(M_2|B)$.