An Introduction
to
Applied Partial Differential Equations

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Problems 1

1. Use the Green's function method to solve the following boundary value problems:

   (a) $-u'' = f(x), \quad 0 < x < 1, \quad u(0) = a, \quad u(1) = b,$

   (b) $-(u'' + u) = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) + u'(1) = 0,$

   (c) $-(u'' + u) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0.$

   (d) $-u'' = x, \quad 0 < x < 1, \quad u(0) = a, \quad u'(1) = b.$

2. Use separation of variables to solve the initial-boundary value problem

   (a)

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = f(x), \tag{2}$$

   i. $f(x) = 3 - 2\cos 4\pi x;$

   ii. $f(x) = \begin{cases} 0, & 0 < x \le \frac{1}{2}, \\ 2x, & \frac{1}{2} < x < 1. \end{cases}$

   (b)

$$u_t(x,t) = u_{xx}(x,t), \quad -1 < x < 1, \quad t > 0, \tag{3}$$

$$u(-1,t) = u(1,t), \quad u_x(-1,t) = u_x(1,t), \tag{4}$$

$$u(x,0) = 3x - 2. \tag{5}$$

3. Consider the problem of heat flow in a uniform rod without internal sources of heat, when the near endpoint is kept at zero temperature and the far endpoint is kept in an open air of zero temperature[1] . That is, solve the corresponding initial-boundary value problem

$$u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < l, \quad t > 0, \quad k > 0, \tag{6}$$

$$u(0,t) = 0, \quad u_x(l,t) + h u(l,t) = 0, \quad u(x,0) = f(x), \tag{7}$$

where $h > 0$.

4. Given the initial value problem

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{8}$$

$$u(x,0) = f(x), \tag{9}$$

where

$$f(x) = \begin{cases} \frac{1}{2\epsilon}, & -\epsilon < x < \epsilon, \\ 0, & |x| > \epsilon, \end{cases} \tag{10}$$

(a) Show that it is equivalent to

$$u_t(x,t) - u_{xx}(x,t) = \delta(t) f(x), \quad -\infty < x < \infty, \quad t > 0, \tag{11}$$

$$u(x,0) = 0. \tag{12}$$

(b) Show that the solution reduces to

$$u(x,t) = \frac{1}{4\epsilon}\left[\operatorname{erf}\left(\frac{x+\epsilon}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-\epsilon}{2\sqrt{t}}\right)\right]. \tag{13}$$

(c) Show that as $\epsilon \to 0$ the result tends to the fundamental solution, as expected.

[1]When one endpoint is in contact with another medium, we use Newton's law of cooling, which states that the heat flux is proportional to the difference in temperatures.

5. Consider the diffusion equation

$$u_t(x,t) = u_{xx}(x,t), \quad 0 \le t < \infty, \tag{14}$$

on the time-dependent domain $ct \le x < \infty$, where c is a constant. We would like to solve the initial-boundary value problem

$$u(x,0) = 0, \quad u(ct,t) = f(t), \tag{15}$$

for $t > 0$ and specified function $f(t)$ which prescribes the solution u at the moving left boundary. Calculate the appropriate Green's function and use it to derive the solution.

6. Find the Green's function and compute the solution of the initial-boundary value problem

$$u_t(x,t) = u_{xx}(x,t) + q(x,t), \quad 0 < x < 1, \quad t > 0, \tag{16}$$

$$u(0,t) = u_x(1,t) = 0, \quad u(x,0) = f(x), \tag{17}$$

when

(a) $q(x,t) = 1, \quad f(x) = x,$

(b) $q(x,t) = t, \quad f(x) = 1.$

7. Consider the homogeneous diffusion equation on the interval $(0,1)$ with the inhomogeneous boundary conditions $u(0,t) = g(t)$, $u(1,t) = h(t)$.

(a) Use the transformation

$$u(x,t) = \omega(x,t) + x[h(t) - g(t)] + g(t), \tag{18}$$

to convert the problem into one for the inhomogeneous equation with the homogeneous boundary conditions.

(b) Use the superposition principle and the Green's function to find the solution.

8. Verify the derivation of the solution in Example 2.7, p.43.

9. Use the Green's function method to find the solution of the boundary value problem

$$\triangle u(x,y) = -\sin \pi x \sin 4\pi y, \quad 0 < x < 1, \quad 0 < y < 1, \tag{19}$$

$$u(x,0) = u(x,1) = 0, \quad u(0,y) = u(1,y) = 0. \tag{20}$$

10. Consider Laplace's equation inside the rectangle: $0 < x < 2, \quad 0 < y < 1$. Suppose that $u_x(0,y) = u_y(x,0) = u_y(x,1) = 0$ and $u_x(2,y) = y+1$. Determine the solution.

11. This problem discusses the fact that under an arbitrary coordinate transformation the Cartesian Laplacian $\triangle u(x,y) \equiv u_{xx} + u_{yy}$ is not necessarily preserved. Consider Laplace'e equation $u_{xx} + u_{yy} = 0$ in some domain D of the xy-plane. Suppose that

$$\zeta = \varphi(x,y), \quad \eta = \psi(x,y), \tag{21}$$

denote a change of coordinates in D. Assume that φ and ψ are both of class C^2 and one-to-one on D.

(a) Show that $U(\zeta,\eta) \equiv u(x(\zeta,\eta), y(\zeta,\eta))$ does not, in general, satisfy the Cartesian form $U_{\zeta\zeta} + U_{\eta\eta} = 0$.

(b) Assume that the transformation of coordinates is such that φ and ψ satisfy the Cauchy-Riemann conditions

$$\varphi_x = \psi_y, \quad \varphi_y = -\psi_x. \tag{22}$$

Conclude that if $u(x,y)$ satisfies the cartesian form of the Laplace's equation so does $U(\zeta,\eta)$.

(c) Discuss the special case of polar coordinates in the plane, where

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}. \tag{23}$$

12. Consider the Dirichlet problem for the two-dimensional Laplacian in the upper-half plane, that is,

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty, \tag{24}$$

with boundary conditions

$$u(x,\infty) = 0, \quad u(x,0) = f(x), \quad f \to 0 \quad \text{as} \quad |x| \to \infty. \tag{25}$$

Derive the solution in the form

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{(x-\xi)^2 + y^2}. \tag{26}$$

Specialize this result to the case when

$$f(x) = \begin{cases} 1, & -a \le x \le b, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

13. Find a solution of the Neumann problem for the Laplace's equation on the rectangle $0 < x < \pi, \quad 0 < y < \pi$, where

$$u_y(x,0) = \cos x - 2\cos^2 x + 1, \quad u_y(x,\pi) = 0, \quad 0 \le x \le \pi, \tag{28}$$

and

$$u_x(0,y) = u_x(\pi,y) = 0, \quad 0 \le y \le \pi. \tag{29}$$

14. Let $f(\theta)$ be a continuous periodic function of period 2π. Define $u(1,\theta) = f(\theta)$ and let (**Poisson Integral Formula**)

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2) f(\theta)}{1 - 2r\cos(\theta - t) + r^2} dt, \quad 0 \le r < 1. \tag{30}$$

One can show that $u(r,\theta)$ is harmonic on the open disc ($r < 1$) and continuous on the closed disk ($r \le 1$). In other words, $u(r,\theta)$ solves the Dirichlet boundary value problem for the Laplace's equation on the unit disk.

Suppose that $u_1(r,\theta)$ and $u_2(r,\theta)$ solve the Dirichlet problem on the unit disk with $u_1(1,\theta) = f_1(\theta)$ and $u_2(1,\theta) = f_2(\theta)$, respectively. Use the Poisson Integral Formula to deduce that

$$\text{if} \quad |f_1(\theta) - f_2(\theta)| < \varepsilon \quad \text{then} \quad |u_1(r,\theta) - u_2(r,\theta)| \le \varepsilon, \quad r < 1. \tag{31}$$

In other words, the harmonic function which is closed on the circle will be closed on the disk. **Hint:** First show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} dt = 1, \quad 0 \le r < 1. \tag{32}$$

Show also that

$$P(r, \theta - t) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \geq 0. \tag{33}$$

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