An Introduction
to
Applied Partial Differential Equations

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Problems 2

1. Find the Fourier series representation of:

   (a)
   $$f(x) = \begin{cases} ax, & -\pi < x \leq 0, \\ bx, & 0 \leq x < \pi. \end{cases} \tag{1}$$

   (b)
   $$f(x) = e^{ax}, \quad -\pi \leq x \leq \pi. \tag{2}$$

   (c)
   $$f(x) = \begin{cases} x+1, & -1 < x < 0, \\ x-1, & 0 < x < 1. \end{cases} \tag{3}$$

   (d)
   $$f(x) = \begin{cases} 0, & -2\pi \leq x < -\pi, \\ \sin x, & -\pi \leq x \leq \pi, \\ 0, & \pi \leq x \leq 2\pi. \end{cases} \tag{4}$$

2. Find the Fourier sine series for $f(x) = |\cos x|$ on $0 \leq x \leq \pi$.

3. Find the Fourier series of $f(x) = \pi^2 - x^2$ on $-\pi \leq x \leq \pi$ and use it to find the Fourier series of x and $x(\pi^2 - x^2)$.

4. Find the Fourier sine and cosine series for the function $f(x) = L - x$ on the interval $[0, L]$.

5. Use the Fourier cosine series on the interval $0 \le x \le l$ to show that

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2. \tag{5}$$

6. (**Bessel's inequality**) Let $f(x)$ be defined on the interval $[-l, l]$, and suppose that $f(x)$ is square integrable on this interval. Assuming that the Fourier series for $f(x)$ exists, show that

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{1}{l}\int_{-l}^{l} |f(x)|^2 dx. \tag{6}$$

7. Use Bessel's inequality for $f(x) = x$ on the interval $[-\pi, \pi]$ to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \le \frac{\pi^2}{6}. \tag{7}$$

8. It can be shown that

$$\sum_{n=1}^{\infty}\frac{\sin nx}{\sqrt{n}} \tag{8}$$

converges for all x to an absolutely integrable function on $[-\pi, \pi]$. Use Bessel's inequality to show that there is no function $f(x)$ on $[-\pi, \pi]$ with

$$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \tag{9}$$

such that the series (8) is its Fourier series.

9. Assuming that $f(x)$ and $f'(x)$ are defined on $[-l, l]$, show that $f'(x)$ is even if $f(x)$ is odd, and vice verse.

10. Show that if the function $f(x)$ is continuous and piecewise C^1 on the interval $[-l, l]$ with $f(-l) = f(l)$, then the Fourier series of $f'(x)$ can be obtained by differentiating term-by-term the Fourier series of $f(x)$. **Hint:** Integrate by parts the Fourier coefficients of $f'(x)$. Note that the integration by parts will not be valid if $f(x)$ is not continuous. What if we do not require that $f(-l) = f(l)$?

11. Find the complex Fourier series of the following functions:

    (a)
    $$f(x) = e^x, \quad -\pi < x < \pi, \tag{10}$$

    (b)
    $$f(x) = \sinh x, \quad -\pi < x < \pi, \tag{11}$$

    (c)
    $$f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 4. \end{cases} \tag{12}$$

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