CHAPTER 1

Boundary Value Problems

In this introductory chapter we discuss the equilibrium equations of one-dimensional continuous systems. These are formulated as boundary value problems for scalar ordinary differential equations. We concentrate on deriving the exact analytical formulae for the solutions of these equations giving us the necessary inside into the physical processes we model.

1.1. Elastic Bar

By a *bar* we mean a finite length one-dimensional continuum that can only be stretched or contracted (deformed, in short) in the longitudinal direction, and is not allowed to bend in a transverse direction. Given a point x on a bar we measure its deformation from the reference position by the *displacement* $u(x)$. That is, a material point which was originally at the position x has been moved to the position $x + u(x)$. We adopt the convention that $u(x) > 0$ means that the material is stretched out, while $u(x) < 0$ describes a contraction by the amount $-u(x)$. We also assume that the left (top) end of the bar is fixed, i.e., $u(0) = 0$.

The internal forces experienced by the bar, known as *stress*, do not necessarily depend on how much the bar is stretched as a whole but rather on how much one material point is moved relative to the neighboring points. This relative amount of elongation is measured by the *strain*. Consider two material points (particles) occupying in the reference configuration positions x and $x + \triangle x$, respectively, where $\triangle x$ is the small section of the bar. When the bar experiences the displacement u the section of length $\triangle x$ gets stretched to the new length

$$[x + \triangle x + u(x + \triangle x)] - [x + u(x)] = \triangle x + [u(x + \triangle x) - u(x)] \,. \qquad (1.1.1)$$

The relative elongation of the segment $\triangle x$ is

$$\frac{u(x+\triangle x)-u(x)}{\triangle x}. \tag{1.1.2}$$

Shrinking the segment of the bar to a point, we obtain the dimensionless strain measure at the position x

$$\varepsilon(x) \equiv \lim_{\triangle x \to 0} \frac{u(x+\triangle x)-u(x)}{\triangle x} = \frac{du}{dx}. \tag{1.1.3}$$

The material the bar is made of is defined by the *constitutive relation* for the stress. This constitutive law tells how the stress depends on the strain when the bar undergoes a deformation. Here we shall only consider a linear relation which, in fact, approximates the real case quite adequately as long as the strain is small. If $\mathfrak{s}(x)$ denotes the stress exerted on a material point which was at the reference position x, we postulate

$$\mathfrak{s}(x) = c(x)\varepsilon(x), \tag{1.1.4}$$

where $c(x)$ measures the *stiffness* of the bar at material point x. If the bar is homogeneous $c(x) = c$ is constant.

We also postulate that the internal stresses of the deformed bar balance the external forces imposed. That is, if $f(x)$ denotes the external force applied at x, we assume that

$$\frac{\mathfrak{s}(x+\triangle x)-\mathfrak{s}(x)}{\triangle x} + \frac{1}{\triangle x}\int_x^{x+\triangle x} f(s)ds = 0 \tag{1.1.5}$$

per unit length of a segment of the bar between positions x and $x + \triangle x$, where $f(x) > 0$ if the bar gets stretched. Invoking the Mean Value Theorem and taking the limit of the left hand side of (1.1.5) as $\triangle x \to 0$ we obtain that

$$f = -\frac{d\mathfrak{s}}{dx} \tag{1.1.6}$$

everywhere along the bar. Substituting the constitutive law (1.1.4) into the equation of balance of forces (1.1.6) and using the definition of the strain function (1.1.3) we obtain the equation of equilibrium

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x), \quad 0 < x < l \tag{1.1.7}$$

for the linearly elastic bar of length (in the reference configuration) l. This is a second-order ordinary differential equation for the displacement $u(x)$. Its general solution depends on two arbitrary constants. They can be uniquely determined by the boundary conditions at the ends of the bar. For example, let

$$u(0) = 0, \quad \mathfrak{s}(l) = c(l)\varepsilon(l) = c(l)u'(l) = 0 \tag{1.1.8}$$

as the left(top) end of the bar is fixed and the other end is assumed to be free (stress free).

EXAMPLE 1.1. Consider a homogeneous bar of unit length subjected to a uniform force, e.g., a bar hanging from a celling and deforming under its own weight. The equilibrium equation (1.1.7) takes the form

$$-c\frac{d^2u}{dx^2} = mg, \tag{1.1.9}$$

where m denotes the mass of the bar and g is the gravitation constant. This is a linear second order equation solution of which is

$$u(x) = -\frac{mg}{2c}x^2 + ax + b. \tag{1.1.10}$$

The arbitrary integration constants a and b can be determined from the boundary conditions (1.1.8). Namely,

$$u(0) = b = 0, \quad u'(1) = -\frac{mg}{2c} + a = 0. \tag{1.1.11}$$

The corresponding unique solution yields the parabolic displacement

$$u(x) = \frac{mg}{c}\left(x - \frac{x^2}{2}\right) \tag{1.1.12}$$

and the linear strain

$$\varepsilon(x) = \frac{mg}{c}(1 - x). \tag{1.1.13}$$

Note that the displacement is maximum at the bottom free end of the bar while strain, and so the stress, are maximum at the fixed end. Note also that as the boundary condition at the free end determines both the strain and the stress at that end the equilibrium equation can be first solved uniquely for stress without calculating the displacement. Such a mechanical configuration is known as *statically determinate*. This is in contrast with the problem in which the displacement is prescribed at both ends of the beam, e.g.,

$$u(0) = 0, \quad u(1) = r. \tag{1.1.14}$$

The general solution to the equilibrium equation (1.1.7) takes the same parabolic form (1.1.10). The unique solution satisfying the boundary conditions (1.1.14) yields

$$u(x) = \frac{mg}{2c}(x - x^2) + rx. \tag{1.1.15}$$

Once the displacement is available we can calculate the stress field

$$\mathfrak{s}(x) = mg(\frac{1}{2} - x) + r. \tag{1.1.16}$$

However, unlike as in case of the bar with the free end, the stress cannot be determined without knowing the displacement. The equilibrium equation (1.1.7) can be re-written in terms of stress

$$-\frac{d\mathfrak{s}}{dx} = mg, \tag{1.1.17}$$

but the integration constant in the general stress solution $\mathfrak{s}(x) = -mgx + a$ cannot be determined as there is no stress boundary condition available. Such a mechanical configuration is called *statically indeterminate*.

Remark 1.2. Our equation (1.1.7) not only describes the mechanical equilibrium of an elastic bar but it models also some other physical systems. For example, this is the thermal equilibrium equation of a bar subjected to an external heat source. Indeed, if $u(x)$ represents the temperature at the position x,

$c(x)$ is the *thermal conductivity* of the material at x, and $f(x)$ denotes the external heat source, then the energy conservation law yields (1.1.7)[1]. A boundary condition $u(l) = r$ corresponds to the situation when an end is kept at a fixed temperature. $u'(l) = 0$, on the other hand, describes a thermally insulated end.

1.2. The Green's Function

The Green's function method is one of the most important approaches to the solution of boundary value problems. It relies on the *superposition principle* for inhomogeneous linear equations. Namely, it builds the general solution out of the solutions to a very particular set of concentrated inhomogeneities.

The superposition principle for a linear homogeneous differential equation states that if $u_1(x)$ and $u_2(x)$ are solutions then every linear combination $\alpha u_1(x) + \beta u_2(x)$ is also a solution, where α and β are arbitrary real numbers. Moreover, if the functions $f_1, \ldots, f_n$ represent the inhomogeneities (forcing terms) of the linear differential equation

$$K[u] = f_i, \quad i = 1, \ldots, n, \tag{1.2.1}$$

where $K[u]$ denotes the differential operator (the left hand side of an equation, e.g., $K[u] = -cu''$), and if $u_1(x), \ldots, u_n(x)$ are the corresponding solutions then the linear *superposition* $\alpha_1 u_1(x) + \alpha_2 u_2(x) \ldots + \alpha_n u_n(x)$ is a solution of

$$K[u] = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n \tag{1.2.2}$$

for any choice of the constants $\alpha_1, \ldots, \alpha_n$.

Our objective here is to use this superposition principle to solve the boundary value problem for a homogeneous elastic bar. To be able to do this we must solve first the boundary value problem with the unit impulse as a source term. Such a solution is called *the Green's function* and it will be used later to construct a solution to the corresponding boundary value problem with an arbitrary forcing term. First, we shall characterize a unit impulse (a point force) concentrated at a point of the bar by introducing the notion of the *delta function*.

[1] See the derivation of the heat conduction equation (2.1.6) in the next chapter.

The Delta Function.

As the impulse is to be concentrated solely at a single point, say y, the delta function $\delta_y(x)$ should be such that

$$\delta_y(x) = 0, \quad \text{for} \quad x \neq y. \tag{1.2.3}$$

Moreover, as we would like the strength of the impulse to be one, and there is no other external force applied, we require that

$$\int_0^l \delta_y(x)dx = 1, \quad \text{as long as} \quad 0 < y < l. \tag{1.2.4}$$

Looking at both conditions $\delta_y(x)$ must satisfy one realizes quickly that there is no such function.

The mathematically correct definition of such a *generalized function*, which can for example be found in [**Ziemer**] (see also [**Lang**]), is well beyond the scope of these notes. It relies on the assumption that for any bounded continuous function $u(x)$

$$\int_0^l \delta_y(x)u(x)dx = u(y), \quad \text{if} \quad 0 < y < l. \tag{1.2.5}$$

Here, we will present the "approximate" definition of the delta function which regards $\delta_y(x)$, considered over the infinite domain $(-\infty, \infty)$, as the limit of a sequence of continuous functions. To this end, let

$$f_n(x; y) \equiv \frac{n}{\pi(1 + n^2(x-y)^2)}. \tag{1.2.6}$$

These functions are such that

$$\int_{-\infty}^{\infty} f_n(x; y)dx = \frac{1}{\pi} \arctan(nx)\Big|_{-\infty}^{\infty} = 1, \tag{1.2.7}$$

and

$$\lim_{n\to\infty} f_n(x; y) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \tag{1.2.8}$$

pointwise, but not uniformly[2]. Hence, we identify $\delta_y(x)$ with the limit

$$\lim_{n\to\infty} f_n(x;y) = \delta_y(x). \tag{1.2.9}$$

Note, however, that this construction of the delta function should only be viewed as a visualization of such a generalized function, and not as its correct mathematical definition. Indeed, within the context of the Riemann's integration theory

$$1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x;y)dx \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x;y)dx = 0 \tag{1.2.10}$$

as the limit of the integral is not necessarily the integral of the limit. On the other hand, this can be made to work if we adopt a somewhat different definition of the limit. This can allow us to justify the formula (1.2.5) as the limit of the approximating integrals. In these notes we will use both definitions of the delta function interchangeably.

Let us consider now the calculus of the delta function, that is, its integration and differentiation. Firstly, assuming that $a < y$ and using the definition (1.2.5) we obtain that

$$\int_a^x \delta_y(s)ds = \sigma_y(x) \equiv \begin{cases} 0, & a < x < y, \\ 1, & x > y > a, \end{cases} \tag{1.2.11}$$

is the *step function*. This is a function which is continuous everywhere except at $x = y$, where its is not defined and experiences a jump discontinuity (1.2.15). The value of the step function at $x = y$ is often left undefined. Motivated by Fourier theory[3] we set $\sigma_y(y) = \frac{1}{2}$. Interestingly enough we obtain the same result using the characterization of the delta function as the limit of the sequence $f_n(x;y)$. Indeed, if

$$g_n(x) \equiv \int_{-\infty}^x f_n(t,0)dt = \frac{1}{\pi}\arctan(nx) + \frac{1}{2}, \tag{1.2.12}$$

then

[2]See Definition B.18.

[3]See Theorem B.4.

$$\lim_{n\to\infty} g_n(x) = \sigma(x) \equiv \sigma_0(x) \tag{1.2.13}$$

pointwise. In tern, the Fundamental Theorem of Calculus allows us to identify the derivative of the step function with the delta function;

$$\frac{d\sigma_y(x)}{dx} = \delta_y(x). \tag{1.2.14}$$

In fact, this enables us to differentiate any discontinuous function having finite jump discontinuities at isolated points. Suppose the function $f(x)$ is differentiable everywhere except at a single point y at which it has a *jump discontinuity*

$$[f(y)] = f_+(y) - f_-(y) \equiv \lim_{x\to y^+} f(x) - \lim_{x\to y^-} f(x). \tag{1.2.15}$$

We can write

$$f(x) = g(x) + [f]\sigma_y(x), \tag{1.2.16}$$

where $g(x) = f(x) - [f]\sigma_y(x)$ is a continuous function, but is not necessarily differentiable at y. Therefore,

$$f'(x) = \begin{cases} g'(x), & x \neq y \\ [f]\delta_y(x), & x = y. \end{cases} \tag{1.2.17}$$

In short, we write

$$f'(x) = g'(x) + [f]\delta_y(x). \tag{1.2.18}$$

Example 1.3. Consider the function

$$f(x) = \begin{cases} -x+1, & x < 0, \\ 0, & 0 < x < 1, \\ x^2, & x > 1. \end{cases} \tag{1.2.19}$$

It has two jump discontinuities: $[f] = -1$ at $x = 0$, and $[f] = 1$ at $x = 1$. Utilizing the construction (1.2.16) one gets that

$$g(x) = f(x) + \sigma(x) - \sigma_1(x) = \begin{cases} -x+1, & x<0, \\ 1, & 0<x<1, \\ x^2, & x>1 \end{cases}$$

and

$$f'(x) = g'(x) - \delta(x) + \delta_1(x) = -\delta(x) + \delta_1(x) + \begin{cases} -1, & x<0, \\ 0, & 0<x<1, \\ 2x, & x>1. \end{cases}$$

To find the derivative $\delta_y'(x)$ of the delta function let us determine its effect on a function $u(x)$ by looking at the limiting integral

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{df_n(x;0)}{dx}u(x)dx = \lim_{n\to\infty} f_n(x;0)u(x)\Big|_{-\infty}^{\infty} - \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x;0)u'(x)dx \tag{1.2.20}$$

$$= -\int_{-\infty}^{\infty}\delta_0(x)u'(x)dx = -u'(0),$$

where the integration by parts was used and where the function $u(x)$ is assume continuously differentiable and bounded to guarantee that

$$\lim_{n\to\infty} f_n(x;0)u(x)|_{-\infty}^{\infty} = 0. \tag{1.2.21}$$

Hence, we postulate that $\delta_y'(x)$ is a generalized function such that

$$\int_0^l \delta_y'(x)u(x)dx = -u'(y). \tag{1.2.22}$$

Note that this definition of the derivative of the delta function is compatible with the formal integration by parts procedure

$$\int_0^l \delta_y'(x)u(x)dx = \delta_y(x)u(x)\Big|_0^l - \int_0^l \delta_y(x)u'(x)dx = -u'(y). \tag{1.2.23}$$

Note also that one may view the derivative $\delta_y'(x)$ as the limit of the sequence of derivatives of the "approximating" functions $f_n(x;0)$. That is,

$$\delta_0'(x) = \lim_{n\to\infty} \frac{df_n(x;0)}{dx} = \lim_{n\to\infty} \frac{-2n^3\pi}{\pi(1+n^2x^2)^2}. \tag{1.2.24}$$

These are interesting rational functions. First of all, it easy to see that the sequence converges pointwise, but not uniformly, to 0. Also, elementary calculations revile, that the graphs of the functions consist of two increasingly concentrated symmetrically positioned at $x = \pm\frac{1}{n\sqrt{3}}$ spikes, and that the amplitudes of these spikes approach $\mp\infty$, respectively, as $n \to \infty$.

The Green's Function.

Once we have familiarized ourselves with the delta function we may try now to solve the boundary value problem for a homogeneous elastic bar with the delta function (unit impulse) as its source term. As we have explained earlier, the main idea behind this approach is to use the superposition principle to obtain the solution for a general external force by putting together the solutions to the impulse problems.

Consider a linearly elastic bar of the reference length l subjected to a unit point force $\delta_y(x)$ applied at position $0 < y < l$. The equation governing such a system (1.1.7) takes the form

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = \delta_y(x), \quad 0 < x < l. \tag{1.2.25}$$

The solution $G(x;y)$ to the boundary value problem associated with (1.2.25) is called the *Green's function* of the problem. To illustrate how such a solution comes about let us consider the following homogeneous boundary value problem

$$-u'' = \delta_y(x), \quad u(0) = u(1) = 0, \tag{1.2.26}$$

for a bar of length $l = 1$ and the stiffness $c = 1$, where $0 < y < l$. Integrating the equation twice we obtain that

$$u(x) = \begin{cases} ax + b, & x \le y, \\ -(x-y) + ax + b, & x \ge y. \end{cases} \tag{1.2.27}$$

Taking into consideration the boundary conditions we have that

$$u(0) = b = 0, \quad \text{and} \quad u(1) = -(1-y) + a + b = 0.$$

This implies that $b = 0$, $a = 1 - y$, and the Green's function of this boundary value problem is

$$G(x;y) = \begin{cases} x(1-y), & x \le y, \\ y(1-x), & x \ge y. \end{cases} \tag{1.2.28}$$

This is a continuous, piecewise differentiable function. Its first derivative experiences a jump of magnitude -1 at $x = y$. In fact, this is a piecewise affine function as its graph consists of straight line segments only. Note also that the Green's function, viewed as a function of two variables, is symmetric in x and y. This symmetry has an interesting physical interpretation that the deformation of the bar measured at position x due to the point force applied at position y is exactly the same as the deformation of the bar at position y due to the concentrated force being applied at position x.

Once we have the Green's function available we can solve the general inhomogeneous problem

$$-u'' = f(x), \quad u(0) = u(1) = 0, \tag{1.2.29}$$

by the linear superposition method. To be able to do this we need first to express the forcing term $f(x)$ as a superposition of point forces (impulses) distributed throughout the bar. The delta function comes handy again as, according to (1.2.5), it enables us to write the external forcing term as

$$f(x) = \int_0^1 f(y)\delta_x(y)dy. \tag{1.2.30}$$

One may interpret the external source f as the superposition of an infinitely many point sources $f(y)\delta_x(y)$ of the amplitude $f(y)$ applied throughout the bar at $0 < x < l$. Re-writing the differential equation (1.2.29) as

$$-u'' = \int_0^1 f(y)\delta_x(y)dy \tag{1.2.31}$$

renders the solution $u(x)$ as a linear combination

$$u(x) = \int_0^1 f(y)G(x;y)dy \tag{1.2.32}$$

of solutions to the unit impulse problems. We can verify by direct computation that the formula (1.2.32) gives us the correct answer to the boundary value problem (1.2.29). Indeed, using the formula for the Green's function (1.2.28) we may write the solution of (1.2.29) as

$$u(x) = \int_0^x (1-x)yf(y)dy + \int_x^1 x(1-y)f(y)dy. \tag{1.2.33}$$

Differentiating it once gives us

$$u'(x) = -\int_0^1 yf(y)dy + \int_x^1 f(y)dy.$$

Differentiating it once again shows that

$$u''(x) = -f(x).$$

For a particular forcing term $f(x)$ it may be easier to solve the problem directly rather than by using the corresponding Green's function. However, the advantage of the Green's function method is that it provides the general framework for any and all inhomogeneous equations with the homogeneous boundary conditions. The case of the inhomogeneous boundary value problem will be discussed in the next chapter.

EXAMPLE 1.4. Consider now a different boundary value problem for a uniform bar of length l. Namely,

$$-cu''(x) = \delta_y(x), \quad u(0) = 0, \quad u'(l) = 0, \tag{1.2.34}$$

where c denotes the elastic constant. This problem models the deformation of the bar with one end fixed and the other end free. Integrating this equation twice, we find the general solution

$$u(x) = -\frac{1}{c}\rho(x-y) + ax + b,$$

where

$$\rho(x-y) \equiv \begin{cases} x-y, & x>y, \\ 0, & x<y, \end{cases} \tag{1.2.35}$$

is called the (first order) *ramp function*. Utilizing the given boundary conditions we find the Green's function for this problem as

$$G(x;y) = \begin{cases} x/c, & x \le y, \\ y/c, & x \ge y. \end{cases} \tag{1.2.36}$$

Again, this function is symmetric and affine. Also, it is continuous but not differentiable at the point of application of the external force $x = y$, where its derivative experiences a $-1/c$ magnitude jump. The formula for the solution of the corresponding boundary value problem for the inhomogeneous equation

$$-cu''(x) = f(x), \quad u(0) = 0, \quad u'(l) = 0, \tag{1.2.37}$$

takes the form

$$u(x) = \int_0^l G(x;y)f(y)dy = \frac{1}{c}\left[\int_0^x xf(y)dy + \int_x^l yf(y)dy\right].$$

1.3. Minimum Principle

In this section we shall discuss how the solution to a boundary value problem is a unique minimizer of the corresponding "energy" functional. This minimization property proves to be particularly significant for the design of numerical techniques such as the finite elements method.

We start by taking a short detour to discuss the concept of an adjoint of a linear operator on an inner product vector space [4] . Let $L : U \to W$ denote a linear operator from the inner product vector space U into another (not necessarily different) inner product vector space W. An *adjoint* of the linear operator L is the operator $L^\star : W \to U$ such that

$$\langle L[u]; w\rangle_W = \langle u; L^\star[u]\rangle_U \quad \text{for all} \quad u \in U, \quad w \in W, \tag{1.3.1}$$

[4] The fundamentals of Inner Product Vector Spaces are reviewed in Appendix A.

where the inner products are evaluated on the respective spaces as signified by the corresponding subscripts. Note that if $U = W = \mathbb{R}^n$, with the standard dot product, and the operator L is represented by a $n \times n$ matrix A, then the adjoint $L^\star$ can be identified with the transpose A^T.

In the context of an equilibrium equation for a one-dimensional continuum the main linear operator is the derivative $D[u] = du/dx$. It operates on the space of all possible displacements U into the space of possible strains W. To evaluate its adjoint we impose on both vector spaces the same standard L^2- inner product, i.e.,

$$\langle u; \tilde{u}\rangle_U \equiv \int_a^b u(x)\tilde{u}(x)dx, \quad \langle w; \tilde{w}\rangle_W \equiv \int_a^b w(x)\tilde{w}(x)dx. \tag{1.3.2}$$

According to (1.3.1) the adjoint $D^\star$ of the operator D must satisfy

$$\langle D[u]; w\rangle_W = \langle \frac{du}{dx}; w\rangle_W = \int_a^b \frac{du}{dx} w(x)dx = \langle u; D^\star[w]\rangle_U = \int_a^b u(x)D^\star[w](x)dx, \tag{1.3.3}$$

for all $u \in U$ and $w \in W$. Note, however, that the integration by part yields

$$\begin{aligned}\langle D[u]; w\rangle_W &= \int_a^b \frac{du}{dx} w(x)dx = [u(b)w(b) - u(a)w(a)] - \int_a^b u(x)\frac{dw}{dx}dx \qquad (1.3.4)\\ &= [u(b)w(b) - u(a)w(a)] + \langle u; \frac{dw}{dx}\rangle_U.\end{aligned}$$

This suggests that

$$\left(\frac{d}{dx}\right)^\star = -\frac{d}{dx}, \tag{1.3.5}$$

provided the functions $u \in U$ and $w \in W$ are such that

$$[u(b)w(b) - u(a)w(a)] = 0. \tag{1.3.6}$$

This will be possible if we impose suitable boundary conditions. In other words, if we define the vector space

$$U = \{u(x) \in C^1[a, b] : \quad u(a) = u(b) = 0\} \tag{1.3.7}$$

as containing all continuously differentiable functions (displacements) vanishing at the boundary, and restrict the operator D to U, the definition of the adjoint (1.3.5) will hold. Obviously, these are not the only boundary conditions guaranteeing (1.3.5). The other possible choice is the space of displacements

$$U = \{u(x) \in C^1[a,b] : \quad u'(a) = u'(b) = 0\}$$

Yet another possibility is the strain space

$$W = \{w(x) \in C^1[a,b] : \quad w(a) = w(b) = 0\}.$$

This is the case when due to the lack of support at the ends the displacement is undetermined and the stresses vanish.

Consider now a homogeneous bar[5] of length l, with material stiffness $c = 1$, and suitable boundary conditions (guaranteeing (1.3.5)). Then, the equilibrium equation (1.1.7) takes the form

$$K[u] = f, \quad \text{where} \quad K = D^\star \circ D = -D^2 \tag{1.3.8}$$

is *self-adjoint*, that is, $K^\star = K$. Indeed,

$$K^\star = (D^\star \circ D)^\star = D^\star \circ D^{\star\star} = D^\star \circ D = K. \tag{1.3.9}$$

Moreover, K is *positive definite*, where a linear operator $K : U \to U$ is said to be positive definite if it is self-adjoint and

$$\langle K[u]; u\rangle_U > 0, \quad \text{for all} \quad 0 \neq u \in U. \tag{1.3.10}$$

To verify the positivity condition for $K = D^\star \circ D$ note that

$$\langle K[u]; u\rangle_U = \langle D^\star[D[u]]; u\rangle_U = \langle D[u]; D[u]\rangle_U = ||D[u]||^2 \geq 0, \tag{1.3.11}$$

and its is positive if and only if $D[u] = 0$ only for $u \equiv 0$. This is not true in general. However, if we impose the homogeneous boundary conditions $u(0) = u(l) = 0$

[5]To use the same framework for the analysis of a bar, or a beam, made of the inhomogeneous material one needs to modify the inner product on the space of strains by introducing, as shown in Section 1.4, a weighted L^2-inner product.

the vanishing of the first derivative of u implies its vanishing everywhere, proving that K is positive-definite. In fact, the same is true for the mixed boundary value problem $u(0) = u'(l) = 0$. However, this is not the case when $u'(0) = u'(l) = 0$ as $D[u] = 0$ for a constant, non-vanishing displacement. The corresponding linear operator K is self-adjoint but not positive-definite.

The minimum principle for the boundary value problem

$$-u'' = f, \quad u(0) = u(l) = 0 \tag{1.3.12}$$

can be formulated as the minimization of the functional

$$\mathfrak{P}[u] = \frac{1}{2}\left|\left|D[u]\right|\right|^2 - \langle u; f\rangle_U = \int_0^l \left[\frac{1}{2}(u'(x))^2 - f(x)u(x)\right] dx \tag{1.3.13}$$

over the space of functions $U = \{u(x) \in C^2[0,l] : \quad u(0) = u(l) = 0\}$.

Remark 1.5. Formally, we seek a function, say $\overline{u}(x)$, from among the functions belonging to the space U, such that

$$\mathfrak{P}[\overline{u}] = \min_{u\in U} \mathfrak{P}[u].$$

Suppose now that $\overline{u}$ is such a minimum and assume that

$$\mathfrak{P}[u] = \int_0^l f(x,u,u')dx.$$

Also, let $w_\epsilon(x) = \overline{u}(x) + \epsilon\eta(x)$ represent a curve of functions in the space U. Note that this implies that the function $\eta(x)$ vanishes at both ends of the interval $[0,l]$. Restricting the functional $\mathfrak{P}$ to the curve w_ϵ consider the real-valued function

$$\mathfrak{i}(\epsilon) = \mathfrak{P}[w_\epsilon].$$

Since $\overline{u}$ is a minimizer of $\mathfrak{P}[\cdot]$ we observe that $\mathfrak{i}[\cdot]$ has a minimum at $\epsilon = 0$. Therefore

$$\mathfrak{i}'(0) = 0.$$

Computing explicitly the derivative we obtain that

$$\mathfrak{i}'(\epsilon) = \int_0^l \left[\frac{\partial f}{\partial w}\eta(x) + \frac{\partial f}{\partial w'}\eta'(x)\right] dx.$$

Integrating the second term by parts and taking into account the boundary conditions for the variation $\eta(x)$, we get that

$$\mathfrak{i}'(\epsilon) = \int_0^l \eta(x) \left[\frac{\partial f}{\partial w} - \frac{d}{dx} \left(\frac{\partial f}{\partial w'} \right) \right] dx.$$

As it is valid for all variations $\eta(x)$, it vanishes when

$$\frac{\partial f}{\partial w} - \frac{d}{dx} \left(\frac{\partial f}{\partial w'} \right) = 0. \tag{1.3.14}$$

This partial differential equation equation is known as the *Euler-Lagrange equation*. Its solution is the minimizer $\overline{u}$. Although any minimizer of $\mathfrak{P}[\cdot]$ is a solution of the corresponding Euler-Lagrange equation, the converse is not necessarily true.

The functional $\mathfrak{P}[u]$ represents the total potential energy of the bar due to the deformation $u(x)$. The first term measures the internal energy due to the strain $u'(x)$ (the *strain energy*) while the second part is the energy due to the external source $f(x)$. The solution to (1.3.12) is the minimizer of $\mathfrak{P}[u]$ over all functions satisfying the given boundary conditions.

EXAMPLE 1.6. To illustrate the importance of the positive-definiteness of the given boundary value problem let us consider the following boundary value problem in strains

$$-u'' = f, \quad u'(0) = u'(l) = 0. \tag{1.3.15}$$

Integrating the equation twice, we find

$$u(x) = ax + b - \int_0^x \left(\int_0^y f(s) ds \right) dy. \tag{1.3.16}$$

Since

$$u'(x) = a - \int_0^x f(s) ds,$$

the boundary condition at $x = 0$ yields $a = 0$. The second boundary condition at $x = l$ implies that

$$u'(l) = \int_0^l f(s)ds = 0. \tag{1.3.17}$$

This is not true in general, unless the source term has the zero mean. But even if the distribution of external forces is such that the mean is zero, the solution

$$u(x) = b - \int_0^x \left(\int_0^y f(s)ds \right) dy \tag{1.3.18}$$

is not unique as the constant b remains unspecified. Physically, this corresponds to an unstable situation. Indeed, if the ends of the bar are left free there exists translation instability in the longitudinal direction.

1.4. Elastic Beam

In this short section we briefly discuss the use (possibly with some necessary adaptation) of the methods developed in the previous sections to analyze the deformation of an elastic (planar) beam. Here by a *beam* we understand a one-dimensional continuum which in addition to being able to stretch in the longitudinal direction is also allowed to bend in a plane, say (x, y). However, to simplify matters, we will only consider that it can bend, neglecting its longitudinal deformations. Consider therefore a beam of a reference length l, and let $y = u(x)$ denote the displacement in the transversal direction. As the beam bends we postulate that its bending moment $\omega(x)$ is proportional to the curvature of the beam

$$\kappa(x) \equiv \frac{u''}{(1 + u'^2)^{3/2}}. \tag{1.4.1}$$

Hence,

$$\omega(x) = c(x)\kappa(x) = \frac{c(x)u''}{(1 + u'^2)^{3/2}}. \tag{1.4.2}$$

If we assume that $u'(x)$ is small, i.e., the beam does not bend too far from its natural straight position, then the curvature is approximately equal to $u''(x)$, and the linearized constitutive relation for the beam assumes the form

$$\omega(x) = c(x)u''(x). \tag{1.4.3}$$

The linearized curvature $\kappa(x) = u''(x)$ plays the role of the (bending) *strain* [**Ogden**].

Relying on the law of balance of moments of forces and using (1.4.3) we obtain the equilibrium equation for the beam as the forth order ordinary differential equation

$$\frac{d^2}{dx^2}\left(c(x)\frac{d^2u}{dx^2}\right) = f(x). \qquad (1.4.4)$$

To be able to determine any particular equilibrium configuration, equation (1.4.4) must be supplemented by a set of boundary conditions. As the equation is of order four we need four boundary conditions; two at each end of the beam. For example, we may assume that $u(0) = \omega(0) = \omega(l) = \omega'(l) = 0$ which describes the situation in which one end of the beam is simply supported while the other is free.

Note that the balance law (1.4.4) can be viewed, with the proper choice of the inner product and the boundary conditions, as written in the adjoint form. To this end, let us consider the differential operator $L \equiv D^2 = D \circ D$, where as before $D \equiv \frac{d}{dx}$. The equilibrium equation (1.4.4) takes the form

$$L[cu''] = f. \qquad (1.4.5)$$

Let us also introduce the weighted inner product

$$\langle \upsilon; \tilde{\upsilon} \rangle \equiv \int_0^l \upsilon(x)\tilde{\upsilon}(x)c(x)dx \qquad (1.4.6)$$

on the space of strains $\upsilon(x) \equiv \kappa(x) = u''(x)$, where $c(x) > 0$. One can easily check that this is indeed an inner product. To compute the adjoint operator $L^\star$ we need to evaluate

$$\langle L[u]; \upsilon \rangle = \int_0^l cL[u]\upsilon dx = \int_0^l c\frac{d^2u}{dx^2}\upsilon dx \qquad (1.4.7)$$

where differentiating by parts twice

$$\int_0^l c\frac{d^2u}{dx^2}\upsilon dx = \left[c\frac{du}{dx}\upsilon - u\frac{d(c\upsilon)}{dx}\right]\Bigg|_0^l + \int_0^l u\frac{d^2(c\upsilon)}{dx^2}dx. \qquad (1.4.8)$$

Consequently, if the functions u and υ are such that

$$\left[c\frac{du}{dx}\upsilon - u\frac{d(c\upsilon)}{dx}\right]\Bigg|_0^l = [u'(l)\omega(l) - u(l)\omega'(l)] - [u'(0)\omega(0) - u(0)\omega'(0)] = 0 \tag{1.4.9}$$

then

$$L^\star[\upsilon] = \frac{d^2}{dx^2}(c\upsilon). \tag{1.4.10}$$

The equilibrium equation (1.4.4) can be written as

$$L^\star[\upsilon] = f, \tag{1.4.11}$$

and there is a quite a variety of possible *self-adjoint boundary conditions* as determined by (1.4.9).

Although the beam operator $L^\star \circ L$ is not self-adjoint it is positive definite for the appropriate boundary conditions, as evident from its form. As usually, the key condition is that D^2 vanishes only on the constant zero function of the appropriate space of functions. Since $D^2[u]$ vanishes if and only if u is affine, the boundary conditions must be such that they force all affine functions to vanish everywhere. For example, having one fixed end ($u(0) = u'(0) = 0$) will be sufficient, while having one simply supported end ($u(0) = \omega(0) = 0$) and one free end ($\omega(l) = \omega'(l) = 0$) will not do.

If the homogeneous boundary conditions are chosen so that the beam operator L is positive definite, it can be shown that the solution to the boundary value problem (1.4.11) is the unique minimizer of the corresponding energy functional

$$\mathfrak{P}[u] = \frac{1}{2}\left|\left|D^2[u]\right|\right|^2 - \langle u; f\rangle = \int_0^l \left[\frac{1}{2}c(x)u''(x) - f(x)u(x)\right] dx. \tag{1.4.12}$$

EXAMPLE 1.7. Consider a uniform beam, with $c(x) \equiv 1$, of the reference length $l = 1$ and such that one end is fixed and the other end is free. In case there are no external forces the equilibrium equation (1.4.4) takes a very simple form

$$\frac{d^4u}{dx^4} = 0. \tag{1.4.13}$$

Its general solution

$$u = ax^3 + bx^2 + cx + d \tag{1.4.14}$$

must satisfy the following boundary conditions:

$$u(0) = u'(0) = \omega(1) = \omega'(1) = 0. \tag{1.4.15}$$

This yields the solution

$$u = \frac{1}{6}x^2(x-3). \tag{1.4.16}$$

To solve the forced beam problem we start by finding the appropriate Green's function. This means that we need to solve first the equation

$$\frac{d^4u}{dx^4} = \delta_y(x) \tag{1.4.17}$$

for the (1.4.15) boundary conditions. Integrating the equation four times and using the fact that the integral of the delta impulse is the ramp function (1.2.35) we obtain the general solution

$$u(x) = ax^3 + bx^2 + cx + d + \begin{cases} \frac{1}{6}(x-y)^3, & x > y, \\ 0, & x < y, \end{cases}. \tag{1.4.18}$$

The boundary conditions imply that

$$u(0) = d = 0, \quad u'(0) = c = 0, \quad \omega(1) = 6a+2b+1-y = 0, \quad \omega'(1) = 6a+1 = 0. \tag{1.4.19}$$

Therefore, the Green's function has the form

$$G(x,y) = \begin{cases} \frac{1}{2}x^2(y - \frac{x}{3}), & x < y, \\ \frac{1}{2}y^2(x - \frac{y}{3}), & x > y. \end{cases} \tag{1.4.20}$$

The Green's function is again symmetric in x and y as the boundary value problem we are dealing with is self-adjoint.

The general solution of the corresponding forced boundary value problem

$$\frac{d^4u}{dx^4} = f(x), \quad u(0) = u'(0) = \omega(1) = \omega'(1) = 0 \tag{1.4.21}$$

is given by the superposition formula

$$u(x) = \int_0^1 G(x,y)f(y)dy = \frac{1}{2}\int_0^x y^2(x - \frac{y}{3})f(y)dy + \frac{1}{2}\int_x^1 x^2(y - \frac{x}{3})f(y)dy. \tag{1.4.22}$$