APPENDIX A

Normed and Inner Product Vector Spaces

A.1. Inner Product Vector Spaces

The concepts of a norm and an inner product on a vector space formalize, and the same time generalize, the Euclidean notions of length and angle. In this chapter we make a brief and necessarily rather sketchy presentation of these concepts, and show how they appear in the context of vector spaces, both finite and infinite-dimensional. We start our presentation by introducing the notion of an inner product on a vector space.

Let V denote a real vector space. An *inner product* on V is a real-valued function

$$\langle\cdot;\cdot\rangle : V \times V \to \mathbb{R} \tag{A.1.1}$$

which is bilinear (linear in each argument), symmetric and positive definite. Being positive definite means that

$$\langle \mathbf{v};\mathbf{v}\rangle \geq 0, \quad \text{for any vector} \quad \mathbf{v} \in V, \tag{A.1.2}$$

and

$$\langle \mathbf{v};\mathbf{v}\rangle = 0 \quad \text{if and only if} \quad \mathbf{v} = \mathbf{0}. \tag{A.1.3}$$

A vector space with a given inner product is called an *inner product vector space*. Be aware that any particular vector space may admit several different inner products.

Example A.1. Consider $V = \mathbb{R}^n$. The standard *Euclidean inner product* is defined as

$$\langle \mathbf{v};\mathbf{w}\rangle = \sum_{i=1}^{n} v_i w_i, \tag{A.1.4}$$

where $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. This is not the only possible inner product on $\mathbb{R}^n$, although most important. For example, let $c_1, \ldots, c_n$ be a set of positive numbers. Define the *weighted inner product*

$$\langle \mathbf{v}; \mathbf{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i. \tag{A.1.5}$$

EXAMPLE A.2. Let $[a, b]$ be a bounded, close interval on $\mathbb{R}$. Consider the vector space $C[a, b]$ of all continuous functions $f : [a, b] \to \mathbb{R}$, and define an inner product as

$$\langle f; g \rangle = \int_a^b f(x)g(x)dx. \tag{A.1.6}$$

Notice that the assumption that all our functions are continuous on $[a, b]$ is essential for the given product to be definite. Relaxing the continuity assumption and extending the selection of functions to include functions which are, for example, piecewise continuous creates problems. Indeed, consider a function h which is zero everywhere on $[a, b]$ except at a single point, at which it is 1. It it obvious that $\langle h; h \rangle = 0$ despite the fact that $h \neq 0$.

The inner product function spaces play essential role in the development of the theory of Fourier series and the solution to the boundary value problems.

Given the inner product vector space V there exists a *norm* associated with the given inner product. Such a norm is defined as the square root of an inner product of a vector with itself:

$$\| \mathbf{v} \| = \sqrt{\langle \mathbf{v}; \mathbf{v} \rangle}. \tag{A.1.7}$$

The positive definiteness of the inner product implies that the norm (A.1.7) is nonnegative, and that it vanishes if and only if $\mathbf{v} \in V$ is the zero vector.

In general, a *norm* on the real vector space V is a non-negative , real-valued function

$$\| \cdot \| : V \to \mathbb{R}^+ \tag{A.1.8}$$

which is homogeneous:

$$\| c\mathbf{v} \| = |c| \, \| \mathbf{v} \|, \quad \text{for any} \quad c \in \mathbb{R}, \quad \text{and} \quad \mathbf{v} \in V, \tag{A.1.9}$$

satisfies *triangle inequality*:

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \| \quad \text{for any} \quad \mathbf{v}, \mathbf{w} \in V, \tag{A.1.10}$$

and is positive definite:

$$\| \mathbf{v} \|= 0 \quad \text{if and only if} \quad \mathbf{v} = 0. \tag{A.1.11}$$

EXAMPLE A.3. The Euclidean inner product (A.1.4) induces on $\mathbb{R}^n$ the norm

$$\| \mathbf{v} \|= \langle \mathbf{v}; \mathbf{v} \rangle = \sqrt{\sum_{i=1}^{n} v_i^2}. \tag{A.1.12}$$

The corresponding weighted norm takes the form

$$\| \mathbf{v} \|= \sqrt{\sum_{i=1}^{n} c_i^2 v_i^2}. \tag{A.1.13}$$

The L^2-norm on $C[a, b]$ is:

$$\| f \|= \sqrt{\int_a^b f(x)^2 dx}. \tag{A.1.14}$$

On of the most important relations between the inner product of two vector and the associated norms of these vectors is the *Cauchy-Schwarz inequality*.

THEOREM A.4. *Every inner product satisfies the Cauchy-Schwarz inequality*

$$|\langle \mathbf{v}; \mathbf{w} \rangle \leq \| \mathbf{v} \| \| \mathbf{w} \|, \quad \mathbf{v}, \mathbf{w} \in V. \tag{A.1.15}$$

Equality holds if and only if the vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel*[1].

PROOF. First, observe that if any of the vectors $\mathbf{v}, \mathbf{w}$ is zero the inequality is trivially satisfied as both sides vanish. We, therefore, assume that both vectors are different from the zero vector. Second, consider the function

$$f(t) =\| \mathbf{v} + t\mathbf{w} \|^2$$

where $t \in \mathbb{R}$ is an arbitrary scalar. Using the bilinearity of the inner product and the definition of the norm (A.1.7) we obtain that

$$f(t) =\| \mathbf{w} \|^2 t^2 + 2\langle \mathbf{v}; \mathbf{w} \rangle t+ \| \mathbf{v} \| .$$

[1]Two vectors are considered parallel if there exists a scalar such that one vector is the multiple of the other. According to this convention the zero vector is parallel to every other vector.

The function $f(t)$ in nonnegative and it attains a minimum as its leading coefficient is positive. The minimum is taken at a point at which the derivative $f'(t)$ vanishes. Namely,

$$f'(t) = 2 \parallel \mathbf{w} \parallel^2 t + 2\langle \mathbf{v}; \mathbf{w} \rangle^2 = 0,$$

when

$$t = -\frac{\langle \mathbf{v}; \mathbf{w} \rangle}{\parallel \mathbf{w} \parallel^2}.$$

Substituting this into the definition of the function we obtain that

$$\parallel \mathbf{v} \parallel^2 \geq \frac{\langle \mathbf{v}; \mathbf{w} \rangle^2}{\parallel \mathbf{w} \parallel^2}, \quad \text{or equivalently} \quad \parallel \mathbf{v} \parallel^2 \parallel \mathbf{w} \parallel^2 \geq \langle \mathbf{v}; \mathbf{w} \rangle^2.$$

Taking the square root of both sides completes the proof. □

Two vectors $\mathbf{v}, \mathbf{w} \in V$, of an inner product vector space V, are called *orthogonal* if $\langle \mathbf{v}; \mathbf{w} \rangle = 0$. In $\mathbb{R}^n$ (with the Euclidean inner product) orthogonality means geometric perpendicularity. In spaces of functions such geometric analogies are not available. Also, being orthogonal with respect to one inner product does not imply being orthogonal with respect to another inner product.

Example A.5. Consider the space $C[0, \pi]$ with the L^2-inner product. There, the function $\sin x$ is orthogonal to $\cos x$. Indeed,

$$\langle \sin x; \cos x \rangle = \int_0^\pi \sin x \cos x dx = \frac{1}{2} \sin^2 x \Big|_0^\pi = 0.$$

Example A.6. Let $P^2[0, 1]$ be the space of all polynomials of degree not bigger than 2 defined on the interval $[0, 1]$. It is elementary to show that the polynomials 1 and x are orthogonal with respect to the standard inner product,

$$\langle a_0 + a_1 x + a_2 x^2; b_0 + b_1 x + b_2 x^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2.$$

However, the same two polynomials are not orthogonal in the L^2-inner product as

$$\langle 1; x \rangle = \int_0^1 x dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}.$$

One of the consequences of the Cauchy-Schwarz inequality is the triangle inequality (A.1.10).

THEOREM A.7. *The norm associated with an inner product satisfies the triangle inequality*

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \| \tag{A.1.16}$$

for every pair of vectors $\mathbf{v}, \mathbf{w} \in V$. *The equality holds if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel.*

PROOF. Consider $\| \mathbf{v}+\mathbf{w} \|^2$ and use the Cauchy-Schwarz inequality (A.1.15) to show that

$$\begin{aligned} \| \mathbf{v} + \mathbf{w} \|^2 &= \| \mathbf{v} \|^2 + 2\langle \mathbf{v}; \mathbf{w} \rangle + \| \mathbf{w} \|^2 \\ &\leq \| \mathbf{v} \|^2 + 2 \| \mathbf{v} \| \| \mathbf{w} \| + \| \mathbf{w} \|^2 = (\| \mathbf{v} \| + \| \mathbf{w} \|)^2 . \end{aligned}$$

Taking the square root of both sides of the inequality completes the proof. □

A.2. Normed Vector Spaces

An inner product vector space has the associated inner product norm, as we showed in the previous section. On the other hand, a vector space may be equipped with a norm which does not come from any inner product.

REMARK A.8. Given a norm in a vector space we may define the notion of a *distance* between vectors:

$$d(\mathbf{v}, \mathbf{w}) = \| \mathbf{v} - \mathbf{w} \| . \tag{A.2.1}$$

Realize that this function possess all the properties we expect from a function measuring distance. It is symmetric, vanishes if and only if $\mathbf{v} = \mathbf{w}$, and satisfies the triangle inequality

$$d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{u}) + d(\mathbf{u}, \mathbf{w}) \tag{A.2.2}$$

for any choice of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

Let us look now at some norms which are not associated with any inner product.

EXAMPLE A.9. Consider again $\mathbb{R}^n$. The 1-*norm* of a vector is defined as:

$$\| \mathbf{v} \|_1 = |v_1| + \cdots + |v_n|. \tag{A.2.3}$$

The p-*norm*, for any integer $p \geq 1$, is

$$\| \mathbf{v} \|_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}. \tag{A.2.4}$$

Taking p to ∞ yields ∞-*norm*

$$\| \mathbf{v} \|_\infty = \sup_{1 \leq i \leq n} \{|v_i|\}. \tag{A.2.5}$$

We leave as an exercise to verify that these are indeed well defined norms[2].

The norms in $\mathbb{R}^n$ have they counterparts in the space of functions.

EXAMPLE A.10. Consider once again $C[a, b]$, the space of all continuous functions on a closed, bounded interval $[a, b]$. The L^p-*norm* is defined as

$$\| f \|_p = \left(\int_a^b |f(x)|^p x \right)^{\frac{1}{p}}. \tag{A.2.6}$$

Respectively, the L^∞-*norm* is defined as

$$\| f \|_\infty = \sup_{a \leq x \leq b} \{|f(x)|\}. \tag{A.2.7}$$

The L^2-norm is the only norm associated with an inner product.

Having all these different norms available one may wounder if they are equivalent in some sense. The answer to this question depends on whether the vector space is finite-dimensional like $\mathbb{R}^n$ or infinite-dimensional like the space $C[a, b]$. That is, any two norms in a finite-dimensional space are equivalent in the following sense:

THEOREM A.11. *Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be any two norms in $\mathbb{R}^n$. Then, there exist positive constants c and C such that*

$$c \| \mathbf{v} \|_1 \leq \| \mathbf{v} \|_2 \leq C \| \mathbf{v} \|_1 \tag{A.2.8}$$

for every $\mathbf{v} \in \mathbb{R}^n$.

The proof of this fact can be found in [**Lang**].

[2] The triangle inequality for the p-norm, known as the *Hölder inequality*, is non-trivial, [**Taylor**].

EXAMPLE A.12. Let us compare the 1-norm and the ∞-norm on $\mathbb{R}^n$. To this end, note first that $\| \mathbf{v} \|_\infty = \sup\{|v_i|\} \leq |v_1| + \cdots + |v_n| = \| \mathbf{v} \|_1$. Thus, we may choose in (A.2.8) $C = 1$. Also, $\| \mathbf{v} \|_\infty = \sup\{|v_i|\} \geq \frac{1}{n}(|v_1| + \cdots + |v_n|) = \frac{1}{n} \| \mathbf{v} \|_1$. Therefore, $c = \frac{1}{n}$ and we have that

$$\frac{1}{n} \| \mathbf{v} \|_1 \leq \| \mathbf{v} \|_\infty \leq \| \mathbf{v} \|_1 \ . \tag{A.2.9}$$

The important consequence of the equivalence of norms in any finite-dimensional vector space, for example $\mathbb{R}^n$, is that the convergence of sequences is norm independent.

We say that the sequence $\{\mathbf{v}^{(k)}\} \subset V$ *converges in the norm* to $\hat{\mathbf{v}} \in V$ if the sequence of scalars $\| \mathbf{v}^{(k)} - \hat{\mathbf{v}} \|$ converges to 0. It is evident from Theorem A.11 that the convergence with respect to one norm implies the convergence with respect to any other norm. Also, the convergence in the norm implies the convergence of the individual components, i.e., $v_i^{(k)} \to \hat{v}_i$. The converse is obviously true as well. This is, in general, not true in infinite-dimensional spaces.

EXAMPLE A.13. Consider the space $C[0,1]$. The sequence of continuous functions

$$f_n(x) = \begin{cases} -nx + 1, & 0 \leq x \leq \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases}$$

is such that its L^∞-norm

$$\| f_n \|_\infty = \sup_{0 \leq x \leq 1} \{|f_n(x)|\} = 1.$$

On the other hand, the L^2-norm is

$$\| f_n \|_2 = \left(\int_0^{\frac{1}{n}} (1 - nx)^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3n}}.$$

It approaches zero when $n \to \infty$. Therefore, there is no constant C such that

$$\| f_n \|_\infty \leq C \| f_n \|_2$$

for this choice of functions, and so for all functions in $C[0,1]$. This proves that the norms L^∞ and L^2 are not equivalent in $C[0,1]$. Notice, however, that

$$\| f \|_2 \leq \| f \|_\infty \tag{A.2.10}$$

for any $f \in C[0,1]$. The convergence in the L^∞-norm implies the convergence in the L^2-norm but not vice versa.

A.3. Complex Vector Spaces

In this section we present a few basic facts about *complex vector spaces* that is the vector spaces in the definition of which the set of real scalars is replaced by the set of complex numbers $\mathbb{C}$. In this set of notes, where we deal with real quantities like the measure of deformation or temperature, we use complex numbers and complex vector spaces primarily to simplify presentation of periodic phenomena. There are, however, physical theories, e.g., quantum mechanics, where complex valued functions are intrinsic as they describe basic physical quantities.

The fundamental example of the complex vector space is the space $\mathbb{C}^n$ consisting of n-tuples of complex numbers $(u_1, u_2, \cdots, u_n)$, where $u_1, \ldots, u_n \in \mathbb{C}$. We can write any vector $\mathbf{u} \in \mathbb{C}^n$ as a linear combination of two real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, namely $\mathbf{u} = \mathbf{x} + i\mathbf{y}$. Its complex conjugate $\overline{\mathbf{u}}$ is obtained by taking the complex conjugate of its coordinates, that is $\overline{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$.

Most vector space concepts curry over from the real case to the complex realm. The only notable exception is the concept of the inner product. Motivated by the desire to have the real associated norm[3] on a complex vector we define the inner product on $\mathbb{C}^n$ as

$$\mathbf{v} \cdot \mathbf{w} = v_1\overline{w}_1 + v_2\overline{w}_2 + \cdots + v_n\overline{w}_n, \tag{A.3.1}$$

where $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$. This construction is known as the *Hermitian inner product* on $\mathbb{C}^n$. For example, if

$$\mathbf{v} = (i, -1), \quad \text{and} \quad \mathbf{w} = (1+i, i),$$

then

$$\mathbf{v} \cdot \mathbf{w} = i(1-i) + (-1)(-i) = 1 + 2i.$$

However,

$$\mathbf{w} \cdot \mathbf{v} = (1+i)(-i) + i(-1) = 1 - 2i,$$

[3]Complex numbers cannot be ordered. Therefore, it does not make any sense to have a complex number $z > 0$.

which shows that the Hermitian dot product is not symmetric. It conjugates, however, under reversal of arguments. Indeed,

$$\mathbf{w} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{w}}, \tag{A.3.2}$$

for any $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$. Also, the dot product is "*sesquilinear*", rather than bilinear, as

$$(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}), \quad \text{while} \quad \mathbf{v} \cdot (c\mathbf{w}) = \overline{c}(\mathbf{v} \cdot \mathbf{w}). \tag{A.3.3}$$

On the other hand, the associated norm looks the same as in the real case:

$$\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{|v_1|^2 + \cdots + |v_n|^2}, \tag{A.3.4}$$

and it is positive for any $\mathbf{v} \neq \mathbf{0}$. $|\cdot|$ denotes here the complex modulus.

The general definition of an inner product on a complex vector space is based on the Hermitian dot product on $\mathbb{C}^n$ and it states that an inner product on a complex vector space V is a complex-valued function $\langle \cdot ; \cdot \rangle : V \times V \to \mathbb{C}$ which is sesquilinear, positive definite and such that

$$\langle \mathbf{v}; \mathbf{w} \rangle = \overline{\langle \mathbf{w}; \mathbf{v} \rangle}$$

for any pair of vectors $\mathbf{v}, \mathbf{w} \in V$.

One can show that the Cauchy-Schwarz inequality (A.1.15), in which the absolute value is replaced by the complex modulus, holds for any inner product complex vector space.

EXAMPLE A.14. Let $\mathbb{C}[-\pi, \pi]$ denote the space of all complex valued continuous function on the interval $[-\pi, \pi] \subset \mathbb{R}$. The Hermitian L^2-inner product is defined as

$$\langle f; g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx. \tag{A.3.5}$$

The associated with this inner product norm is

$$\| f \| = \left(\int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{\frac{1}{2}}. \tag{A.3.6}$$

For example, the set of complex exponential e^{ikx}, where k is any integer, is an orthonormal system of functions. Indeed,

$$\langle e^{ikx}; e^{imx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-m)x} dx = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \tag{A.3.7}$$

where we utilized the fact that $e^{ik\pi} = \left(e^{i\pi}\right)^k = (\cos \pi + i \sin \pi)^k = (-1)^k$. The orthogonality property (A.3.7) of complex exponentials is of significants in the Fourier theory, as we will see in the next chapter.

APPENDIX B

Fourier Theory

This chapter serves as a brief introduction to the Fourier theory and related topics.

B.1. Fourier Series

The basic object of study of the Fourier theory is an infinite trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx\right], \tag{B.1.1}$$

called a *Fourier series*. In general, such a series may not converge without additional assumptions about its coefficients. We would like to know if a Fourier series can converge to a function, and whether a function can be represented by a Fourier series.

A *Fourier series of the function* $f(x)$ is a Fourier series (B.1.1) where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx, \qquad k = 0, 1, 2, \ldots, \tag{B.1.2a}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx, \qquad k = 1, 2, 3, \ldots, \tag{B.1.2b}$$

and where we assume that the integrals are well defined. In fact, as we will see later, these integrals are well defined for a quite a broad class of functions $f(x)$. This obviously does not guarantee the convergence of the series, and certainly not to the function $f(x)$. The choice of the coefficients is, however, dictated by our desire to be able to represent a function by a Fourier series. Indeed, suppose that $f(x)$ is well approximated over the interval $[-\pi, \pi]$ by a Fourier series, that is

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx\right] \tag{B.1.3}$$

for some choice of the coefficients a_k and b_k. Multiply both sides of (B.1.3) by $\cos lx$ and use integration by parts to show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos kx \cos lx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx \cos lx dx = 0 \quad \text{if} \quad k \neq l,, \tag{B.1.4a}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 lx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 lx dx = 1. \tag{B.1.4b}$$

This implies immediately the choice of the Fourier coefficients (B.1.2).

Example B.1. Consider the function $f(x) = x$. Computing its Fourier coefficients directly we obtain that:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0, \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos kx dx = 0, \tag{B.1.5}$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx dx = \frac{1}{\pi}\left[-\frac{x\cos kx}{k} + \frac{\sin kx}{k^2}\right]|_{-\pi}^{\pi} = \frac{2}{k}(-1)^{k+1}. \tag{B.1.6}$$

The vanishing of the coefficients a_k is a consequence of the fact that the function x is odd while $\cos kx$ is an even function. Therefore, the Fourier series of $f(x) = x$ is

$$2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}. \tag{B.1.7}$$

It is not an elementary exercise to determine the convergence of this series. But, even if we overcome this obstacle and determine that it converges, one does not know what it converges to. For example, it certainly does not converge to $f(x) = x$. Indeed, at $x = \pi$ the series (B.1.7) converges to 0, as every term in the series vanishes at π, while $f(\pi) = \pi \neq 0$.

As our example shows the convergence of Fourier series is not a simple matter. The standard tests used to analyze the convergence of power series fail. Also, power series always converge at least at 0 or on an interval (possibly infinite) centered at 0. Fourier series, on the other hand, may converge on rather unusual sets. Moreover, as all components of a Fourier series are 2π periodic the series will, in general, converge to 2π periodic functions. The power series and the Fourier series differ also by what they converge to. Indeed, if a power series converges on an interval it converges to an infinitely differentiable function as all its derivative series converge to the corresponding derivatives. Such functions are called *analytic*. A Fourier series, on the other hand, may converge to discontinuous functions and even to such a "bizarre" function as the Dirac delta function.

We will look first at the issue of periodicity of the limit of a Fourier series. Given a function $f(x)$ for $-\pi < x \leq \pi$ let us define its *periodic extension* as a function $\tilde{f}(x)$ defined everywhere and such that $\tilde{f}(x+2\pi) = \tilde{f}(x)$, and $\tilde{f}(x) = f(x)$ for $-\pi < x \leq \pi$. To this end, let $x \in \mathbb{R}$. Thus, there exists a unique integer m such that $(2m-1)\pi < x \leq (2m+1)\pi$. We therefore postulate that

$$\tilde{f}(x) = f(x - 2m\pi) \qquad \text{for} \qquad (2m-1)\pi < x \leq (2m+1)\pi.$$

Such a function is obviously unique, 2π periodic, and coincides with $f(x)$ on $-\pi < x \leq \pi$. In most cases such a periodic extension is not a continuous function. It is, however, "piecewise continuous".

Definition B.2. A function $f(x)$ is called *piecewise continuous* on the interval $[a, b]$ if it is defined and continuous except possibly a finite number of point. Moreover, at each "point of discontinuity", say x_i, the left and the right hand limits

$$f_+(x_i) \equiv \lim_{x \to x_i^+} f(x), \qquad f_-(x_i) \equiv \lim_{x \to x_i^-} f(x),$$

exist. A function defined on all of $\mathbb{R}$ is piecewise continuous if it is piecewise continuous on every bounded interval.

Note that according to this definition a piecewise discontinuous function may not be defined at a "point of discontinuity". Even if it is defined its value $f(x_i)$ is not necessarily equal to either right or left hand limit. Such a point x_i is called a *jump discontinuity* and we say that the function $f(x)$ experiences a jump of the magnitude

$$[f(x_i)] \equiv f_+(x_i) - f_-(x_i). \tag{B.1.8}$$

If the function $f(x)$ is piecewise continuous its (formal) Fourier series is well defined as the function is integrable on any closed interval. The convergence of such a series, even if the function is continuous on $[-\pi, \pi]$ interval, is still not guaranteed.

DEFINITION B.3. A function $f(x)$ is called *piecewise* C^1 on the interval $[a, b]$ if it is piecewise continuous on $[a, b]$, and continuously differentiable on $[a, b]$ except possibly a finite number of points. If x_i is such a point then the left and right hand limits

$$f'_+(x_i) \equiv \lim_{x \to x_i^+} f'(x), \qquad f'_-(x_i) \equiv \lim_{x \to x_i^-} f'(x), \tag{B.1.9}$$

exist.

Note that a piecewise C^1 function may have points at which either the function experiences a jump but the left and right hand derivatives exist or it is continuous but its derivative experiences a jump. The periodic extension of the function $f(x) = x$ has jump discontinuities of the first type at $\pi + 2m\pi$ while the absolute value $|x|$ is continuous everywhere but its derivative experiences a jump at 0. We are now in the position to state the convergence theorem for Fourier series. The proof will be presented in Section B.4.

THEOREM B.4. *Let $\tilde{f}(x)$ be a periodic extension of a piecewise C^1 function on $[-\pi, \pi]$, then its Fourier series converges at all x to:*

$$\tilde{f}(x) \qquad \textit{whenever } \tilde{f} \textit{ is continuous,}$$

$$\frac{1}{2}\left[\tilde{f}_+(x) + \tilde{f}_-(x)\right] \qquad \textit{if } x \textit{ is a point of discontinuity.}$$

In other words, if in the definition of the periodic extension we replace the values at any discontinuity point x by

$$\tilde{f}(x) = \frac{1}{2}\left[\tilde{f}_+(x) + \tilde{f}_-(x)\right],$$

the Fourier series of a piecewise C^1 periodic extension $\tilde{f}(x)$ will converge to $\tilde{f}(x)$ everywhere.

Example B.5. Let us revisit example B.1, but let the periodic extension of $f(x) = x$ be such that

$$\tilde{f}((2m+1)\pi) = 0, \qquad \text{for any integer m.}$$

Then the Fourier series

$$2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$$

converges everywhere to the "new" periodic extension, and the convergence problem we previously encountered at π is solved. Indeed, for any $x = (2m+1)\pi$ all terms in our Fourier series vanish.

Example B.6. Let $f(x) = |x|$. Observe that as $|x|$ is an even function its periodic extension is continuous. Its Fourier coefficients can easily be computed:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx = \frac{2}{\pi}\int_0^{\pi} xdx = \pi,$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x\cos xdx = -\frac{4}{k^2\pi}, \quad \text{if } k \text{ is odd, and otherwise 0.}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin xdx = 0,$$

as the function $|x|\sin x$ is odd. Therefore, the Fourier series of $f(x) = |x|$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)}{(2k+1)^2}. \tag{B.1.10}$$

According to Theorem B.4, it converges to the periodic extension of $f(x) = |x|$. In particular, if $x = 0$ we obtain that

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}. \tag{B.1.11}$$

This series can be used to obtain an approximation to the number π.

As we have noted in Examples B.5 and B.6 the coefficients of the Fourier cosine series of the function $f(x) = x$, and the sine coefficients of the function $f(x) = |x|$, are 0. This is not a coincidence, but rather a consequence of the fact that x is an odd function while $|x|$ is an even function. Indeed, the following proposition, whose proof is elementary and will be left to the reader, generalizes our observations.

PROPOSITION B.7. *If $f(x)$ is an even function then its Fourier series sine coefficients b_k all vanish, and f can be represented by a Fourier cosine series*

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx. \tag{B.1.12}$$

If $f(x)$ is odd, then its Fourier series cosine coefficients a_k all vanish. Thus, f can be represented by a Fourier sine series

$$f(x) \cong \sum_{k=1}^{\infty} a_k \sin kx. \tag{B.1.13}$$

Conversely, a convergent Fourier cosine (sine) series always represents an even (odd) function.

B.2. Differentiation and Integration of Fourier Series

Knowing that power series can be differentiated and integrated term by term, and that these two operations do not change (except for the end points of the interval of convergence) the convergence of these series, it make sense to investigate whether a similar property holds for Fourier series. The main difference between these two cases is that the power series converge to analytic functions, and hence can be freely differentiated and integrated, while the Fourier series may converge to functions of very different degrees of smoothness. Thus, investigating differentiation and integration of Fourier series we must pay careful attention to the regularity of its limits.

Integration. When attempting to integrate a Fourier series we are faced with the fundamental problem that, in general, the integral of the periodic function is not periodic. However, a Fourier series consists mostly of sine and cosine terms which when integrated are also period functions. The only term we are evidently going to have a problem with is the free term $a_0/2$. Hence, we will first try to integrate Fourier series with the zero constant term:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0.$$

This shows that a function has no constant term in its Fourier series if its *average* on the interval $[-\pi, \pi]$ is zero. The periodic zero average functions are the once that remain periodic upon integration. Indeed, using the definition of a periodic function one can easily confirm the following property:

LEMMA B.8. *Let $f(x)$ be 2π periodic. Then, the integral*

$$g(x) \equiv \int_0^x f(x)dx$$

is a 2π periodic function if and only if f has zero average on the interval $[-\pi, \pi]$.

Furthermore, simple integration by parts shows that:

THEOREM B.9. *If $f(x)$ is a piecewise continuous, 2π periodic function, and has zero average, then its Fourier series can be integrated term by term to produce the Fourier series of*

$$g(x) = \int_0^x f(y)dy \cong m + \sum_{k=1}^{\infty} \left[\frac{a_k}{k} \sin kx - \frac{b_k}{k} \cos kx \right], \tag{B.2.1}$$

where

$$m \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)dx$$

is the average value of $g(x)$.

Note that integrating the Fourier series (B.2.1) on the interval $[-\pi, \pi]$, and observing that the average of any odd function is zero we obtain that

$$m = \sum_{k=1}^{\infty} \frac{b_k}{k}. \tag{B.2.2}$$

This provides a convenient alternative derivation of the sine coefficients of a Fourier series.

EXAMPLE B.10. Let us consider again the function $f(x) = x$. This function is odd hence, has zero average. Integrating its Fourier series (B.1.7) from 0 to x we obtain the series

$$2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} (1 - \cos kx).$$

The constant term of this series is the average of $x^2/2$. As in (B.2.2):

$$m = \sum_{k=1}^{\infty} \frac{b_k}{k} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2} dx = \frac{\pi^2}{6}.$$

This immediately yields the Fourier series of the function x^2:

$$x^2 \cong \frac{\pi^2}{3} - 4\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos kx}{k^2}.$$

As x^2 is an even function its Fourier series converges everywhere to its periodic extension.

Differentiation. As much as integration is making functions nicer, differentiation does the opposite. Therefore, in order to secure the convergence of the derivative of a Fourier series we must start with the sufficiently nice function; but how nice. If the Theorem B.4 is to be applicable the derivative $f'(x)$ must be piecewise C^1. This means that the function $f(x)$ must be at least continuous and piecewise C^2.

THEOREM B.11. *If $f(x)$ is continuous, piecewise C^2 and 2π periodic, then the term by term differentiation of its Fourier series produces the Fourier series for the derivative*

$$g(x) = f'(x) \cong \sum_{k=1}^{\infty} k\left[b_k \cos kx - a_k \sin kx\right]. \tag{B.2.3}$$

EXAMPLE B.12. Consider again $f(x) = |x|$. Differentiating its Fourier series (B.1.10) we obtain

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx}{k}. \tag{B.2.4}$$

On the other hand, the derivative of the absolute value $|x|$ can be represented as a difference of two step functions

$$\sigma(x) - \sigma(-x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

A simple calculation shows that the Fourier series of the step function $\sigma(x)$ is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx}{k}. \tag{B.2.5}$$

It is now easy to see that the derivative of the Fourier series of $|x|$ is indeed the Fourier series of the difference $\sigma(x)-\sigma(-x)$. Moreover, according to Theorem B.4 it converges to it.

REMARK B.13. So far, we have defined a Fourier series on the interval $[-\pi,\pi]$ only. This procedure can obviously be easily adopted to any other 2π length interval. In many applications the functions we deal with are defined on intervals of other lengths. Therefore, it would help to show how the formulas we developed so far change if we change the length of the interval. First, let us note that any symmetric interval $[-l,l]$ can be rescaled to the interval $[-\pi,\pi]$ by the simple change of variables

$$y = \frac{\pi}{l}x.$$

The rescaled function

$$\widehat{f}(y) \equiv f\left(\frac{l}{\pi}y\right),$$

which lives on $[-\pi,\pi]$ interval, has the standard Fourier series

$$\widehat{f}(y) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos ky + b_k \sin ky\right].$$

Going back to the original variable x we deduce that

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l}\right], \tag{B.2.6}$$

where, thanks to the our change of variables, the Fourier coefficients a_k and b_k have the modified formulas

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}dx, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}dx. \tag{B.2.7}$$

According to our convergence Theorem B.4 the given Fourier series converges to the $2l$ periodic extension of $f(x)$ with the midpoint values at jump discontinuities, provided $f(x)$ is piecewise C^1 on $[-l,l]$.

If the function $f(x)$ is defined on an arbitrary interval $[a,b]$ then the first step in the process of developing its Fourier series is to rescale it into a symmetric interval. This can easily be done by the translation

$$y = x - \frac{1}{2}(a+b).$$

The new interval $\left[\frac{1}{2}(a-b), \frac{1}{2}(b-a)\right]$ is symmetric of length $b-a$, and the latter procedure can be utilized.

B.3. Complex Fourier Series and the Delta Function

A natural, and often very convenient, generalization of Fourier series is its complex counterpart, where exponential functions with complex exponents replace sine and cosine functions. Indeed, using Euler's formula

$$e^{\mathrm{i}kx} = \cos kx + \mathrm{i}\sin kx, \tag{B.3.1}$$

we can represent trigonometric functions sine and cosine as

$$\cos kx = \frac{e^{\mathrm{i}kx} + e^{-\mathrm{i}kx}}{2}, \quad \text{and} \quad \sin kx = \frac{e^{\mathrm{i}kx} - e^{-\mathrm{i}kx}}{2\mathrm{i}}. \tag{B.3.2}$$

Substituting these relations into a Fourier series of a piecewise continuous (real or complex valued) function we obtain a new Fourier series representation

$$\sum_{k=1}^{\infty} c_{-k} e^{-\mathrm{i}kx} + \sum_{k=0}^{\infty} c_k e^{\mathrm{i}kx} = \sum_{k=-\infty}^{k=\infty} c_k e^{\mathrm{i}kx}, \tag{B.3.3}$$

where

$$c_k = \frac{1}{2}(a_k + \mathrm{i}b_k), \qquad \text{if} \quad k \le 0, \tag{B.3.4a}$$

and

$$c_k = \frac{1}{2}(a_k - \mathrm{i}b_k), \qquad \text{if} \quad k \ge 0. \tag{B.3.4b}$$

This and Euler's formula (B.3.1) show, in fact, that the *complex Fourier coefficients* can be evaluated directly as

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-\mathrm{i}kx} dx. \tag{B.3.5}$$

Remark B.14. This result is deeper than it seems. It is a consequence (as in the real case) of the *orthonormality* of the complex exponential functions with respect to the Hermitian inner product

$$\langle f; g\rangle \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)} dx, \tag{B.3.6}$$

where $\overline{g(x)}$ denotes the conjugate. Indeed, evoking Euler's formula (B.3.1) one can easily show that

$$\langle e^{\mathrm{i}kx}; e^{\mathrm{i}lx} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(k-l)x} dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases} \tag{B.3.7}$$

Therefore, multiplying the complex Fourier series (B.3.3) by $e^{\mathrm{i}lx}$ and integrating term by term we obtain the formula for the complex Fourier coefficients (B.3.5).

Example B.15. Let us develop the complex Fourier series for the function $f(x) = x$. Its complex Fourier coefficients are

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-\mathrm{i}kx} dx = \frac{(-1)^{k+1}}{\mathrm{i}k} = \frac{(-1)^k i}{k}.$$

Consequently, the complex Fourier series of $f(x) = x$ is

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k i}{k} e^{\mathrm{i}kx}.$$

The reader is asked to show, using once again Euler's formula (B.3.1), that this is exactly the same series as the sine Fourier series (B.1.7).

Example B.16 (**Dirac delta function**). In this example we shall investigate a Fourier series representation of the *delta function* to prove that a Fourier series can converge to a *generalized function*, i.e., a function which although is a limit of a sequence of piecewise continuous functions is not a standard function itself. This example shows also the benefits of using complex Fourier series rather than its real counterpart. First, let us compute complex Fourier coefficients:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x) e^{-\mathrm{i}kx} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x) e^{-\mathrm{i}kx} dx = \frac{1}{2\pi} e^{-\mathrm{i}k0} = \frac{1}{2\pi}. \tag{B.3.8}$$

Therefore, the complex Fourier series of the delta function has the form

$$\frac{1}{2\pi}\sum_{k=-\infty}^{k=\infty} e^{\mathrm{i}kx}. \tag{B.3.9}$$

Observe that this series has the form of an infinite (in both directions) geometric series with the ratio $r = e^{\mathrm{i}x}$. This is in contrast to the real Fourier series of the delta function which, as easily deduced directly from its definition, is

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos kx. \tag{B.3.10}$$

These are obviously two different representations of the same series. However, in order to show that such a series converges to the delta function we will use the complex version.

The complex Fourier series of the delta function (B.3.9) has a form of the geometric series, as we have indicated earlier. In order to determine its convergence let us consider its n$^{\text{th}}$ partial sum

$$s_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{k=n} e^{\mathrm{i}kx}.$$

This is the $2n{+}1$ partial sum of the geometric series with the initial term $a = e^{-\mathrm{i}nx}$ and the ratio $r = e^{\mathrm{i}x}$. It can be therefore computed exactly that

$$\begin{aligned}
s_n(x) &= \frac{1}{2\pi}\sum_{k=-n}^{k=n} e^{\mathrm{i}kx} = \frac{1}{2\pi}e^{-\mathrm{i}nx}\left[\frac{e^{\mathrm{i}(2n+1)x}-1}{e^{\mathrm{i}x}-1}\right] = \frac{1}{2\pi}\left[\frac{e^{\mathrm{i}(n+1)x}-e^{-\mathrm{i}nx}}{e^{\mathrm{i}x}-1}\right] \\
&= \frac{1}{2\pi}\left[\frac{e^{-\mathrm{i}\frac{x}{2}}(e^{\mathrm{i}(n+1)x}-e^{-\mathrm{i}nx})}{e^{-\mathrm{i}\frac{x}{2}}(e^{\mathrm{i}x}-1)}\right] = \frac{1}{2\pi}\left[\frac{e^{\mathrm{i}(n+\frac{1}{2})x}-e^{-\mathrm{i}(n+\frac{1}{2})x}}{e^{\mathrm{i}\frac{x}{2}}-e^{-\mathrm{i}\frac{x}{2}}}\right] \\
&= \frac{1}{2\pi}\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}},
\end{aligned} \tag{B.3.11}$$

where the representation (B.3.2) of the sine function in terms of complex exponentials was utilized. The sequence of partial sums $s_n(x)$ converges at $x = 0$ to infinity, as easily attested from the (B.3.9). Moreover,

$$\int_{-\pi}^{\pi} s_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{k=n} e^{\mathrm{i}kx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{k=n}\left[\cos kx + \mathrm{i}\sin kx\right]dx = 1 \tag{B.3.12}$$

as required for the convergence to the delta function. At any other point x the sequence $s_n(x)$ does not converge to zero. In fact, one can see from (B.3.11) that it oscillates faster and faster. It appears, however, when added up over a large domain, that the oscillations cancel out. The Fourier series of the delta function

$\delta(x)$ does not converge to its periodic extension $\widetilde{\delta}(x)$ point-wise. It may be shown that it converges in a *weak sense*, i.e., in the sense of some integral average.

B.4. Convergence of the Fourier Series

In this section we present basic convergence results for Fourier series. A proper, although inevitably limited, presentation of these fundamental results requires some familiarity with the concepts and methods of advanced mathematical analysis. In our effort to make this presentation as self contained as possible we introduce the necessary tools as they become necessary for the analysis of the convergence of Fourier series. We start our presentation by analyzing the *pointwise convergence* of a Fourier series.

Pointwise Convergence.

Proof of Theorem B.4. Our objective is to show that, given a 2π periodic piecewise C^1 function $f(x)$, the limit of the sequence of partial Fourier sums $s_n(x)$ is the arithmetic average of the left hand and the right hand limits of this function, i.e.,

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}\left[f_+(x) + f_-(x)\right].$$

Consider

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{\mathrm{i}kx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-\mathrm{i}ky} dy\right) e^{\mathrm{i}kx}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{\mathrm{i}k(x-y)} dy,$$

where the formula (B.3.5) for the complex Fourier coefficients was used. Using the summation formula (B.3.11) and the periodicity of the functions involved we calculate that

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin(n+\frac{1}{2})(x-y)}{\sin\frac{1}{2}(x-y)}dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy,$$

where to obtain the last integral we changed the variables from y to $x+y$. If we could now show that

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{0}^{\pi} f(x+y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy = f_+(x), \tag{B.4.1a}$$

and

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{0} f(x+y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy = f_-(x) \tag{B.4.1b}$$

the proof would be complete. To this end, recalling (B.3.12) allows us to claim that

$$\frac{1}{\pi}\int_{0}^{\pi}\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy = 1.$$

Consequently, the statement (B.4.1a) can be replaced by the equivalent statement

$$\lim_{n\to\infty}\int_0^{\pi}\frac{f(x+y)-f_+(x)}{\sin\frac{y}{2}}\sin(n+\frac{1}{2})ydy = 0. \tag{B.4.2}$$

Let

$$g(y) \equiv \frac{f(x+y)-f_+(x)}{\sin\frac{y}{2}}. \tag{B.4.3}$$

Then, using the trigonometric identity for $\sin(n+\frac{1}{2})$, and *Riemann's Lemma*[1] we notice that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^{\pi}\frac{f(x+y)-f_+(x)}{\sin\frac{y}{2}}\sin(n+\frac{1}{2})ydy\\ &=\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\Big(g(y)\sin\frac{y}{2}\Big)\cos nydy+\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\Big(g(y)\cos\frac{y}{2}\Big)\sin nydy = 0\end{aligned}$$

as long as the function $g(y)\sin\frac{y}{2}$ is piecewise continuous. This completes our proof provided we can show that $g(y)$ is piecewise continuous on $[0,\pi]$. Looking at the definition of the function $g(y)$ we can easily see that this is true except possibly at $y=0$. However,

$$\lim_{y\to 0^+} g(y) = \lim_{y\to 0^+}\frac{f(x+y)-f_+(x)}{\frac{y}{2}}\frac{\frac{y}{2}}{\sin\frac{y}{2}} = 2f'_+(x),$$

as the function $f(x)$ is piecewise C^1. This fact confirms that $g(y)$ is piecewise continuous. Identical arguments prove (B.4.1b). □

[1] Riemann's Lemma states that if a function $f(x)$ is piecewise continuous on $[a,b]$ then

$$\lim_{\alpha\to\infty}\int_a^b f(x)\cos\alpha x dx = 0,$$

and similarly if cosine is replaced by sine. For the proof of this fact see for example [**Lang**].

Uniform Convergence. Let us consider a finite-dimensional vector space V, e.g., $\mathbb{R}^n$. In fact, all finite-dimensional spaces of the same dimension are essentially identical (*isomorphic*), as we stated earlier. In order to be able to talk about convergence of sequences in vector spaces we need a measure of a distance between vectors. Hence, let V be a normed vector space with the norm $\|\cdot\| : V \to \mathbb{R}$ (A.1.8).

DEFINITION B.17. Let W be an normed vector space. We say that a sequence $\mathbf{v}_n \in$ W converges in the norm to $\mathbf{w} \in$ W if $||\mathbf{v}_n - \mathbf{w}|| \to 0$ as $n \to \infty$.

In a finite-dimensional vector space this is equivalent to a convergence of individual components of $\mathbf{u}^k = (u_1^k, \ldots, u_n^k)$ to the corresponding components of $\mathbf{u} = (u_1, \ldots, u_n)$. This is one way of saying that in a finite-dimensional space all norms are equivalent (see Theorem A.11), i.e., a convergence in one norm quarantines the convergence in any other norm. This, however, is not true in general in the infinite-dimensional vector spaces. All because of the fact that there are many, not necessarily equivalent, norms in such spaces - Example A.13. Also, in infinite dimensional vector spaces the convergence in the norm does not imply - as we will see later - *pointwise convergence*

$$\lim_{k\to\infty} u_i^k(x) = u_i(x), \quad i = 1, \ldots, \infty \quad \text{for all} \quad x, \tag{B.4.4}$$

of the sequence of values of the functions $u_i^k(x)$.

In addition to the convergence in the norm and the pointwise convergence mechanisms there exists yet another form of convergence; the *uniform convergence*.

DEFINITION B.18. A sequence of real valued function $f_k(x)$ converges *uniformly* to a function $f(x)$ on an interval $I \subset \mathbb{R}$ if, given $\epsilon > 0$ there exists an integer n such that

$$|f_k(x) - f(x)| < \epsilon$$

for all $x \in I$ and all $k \geq n$.

The name uniform converges is a reflection of the fact that the sequence of functions converges "the same way" at all points of the interval I. The choice of the integer n depends only on ϵ and not on a point $x \in I$. Note also, that although uniform convergence implies pointwise convergence the converse may not be true. For example, consider the sequence of functions $f_k(x) = \frac{1}{kx}$ on the interval $(0,1)$. The sequence converges pointwise to $f(x) \equiv 0$ at all points $x \in (0,1)$ but it does not converge uniformly on $(0,1)$ as given ϵ there is no possible choice of the integer n. One can always select a point x so close to 0 that $\frac{1}{nx}$ is larger than ϵ.

The greatest advantage of uniform convergence is that it preserves continuity.

THEOREM B.19. *If a sequence of continuous functions $f_k(x)$ converges uniformly on I then the limit $f(x)$ is a continuous function on I.*

As we are interested here primarily in the convergence of series the following test proves to be particularly useful:

THEOREM B.20 (Weierstrass test). *Suppose that the function $f_k(x)$ are bounded, i.e.,*

$$|f_k(x)| \leq m_k \quad \textit{for all} \quad x \in I,$$

where m_k are positive constants. If, in addition, the series

$$\sum_{k=1}^{\infty} m_k$$

converges, then the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges uniformly on the interval I to a function $f(x)$. In particular, if the partial sums $s_n(x)$ are continuous functions so is the limit $f(x)$.

We advise the reader to prove that one can integrate and differentiate a uniformly convergent series term by term to obtain a uniformly convergent series, provided the series of derivatives is also uniformly convergent.

PROPOSITION B.21. *If the series*

$$\sum_{k=1}^{\infty} f_k(x) = f(x)$$

is uniformly convergent then

$$\int_a^x \left(\sum_{k=1}^{\infty} f_k(y) \right) dy = \sum_{k=1}^{\infty} \int_a^x f_k(y) dy = \int_a^x f(y) dy$$

is uniformly convergent. Also, if the series of derivatives

$$\sum_{k=1}^{\infty} f_k'(x) = g(x)$$

is uniformly convergent then the original series converges uniformly and $g(x) = f'(x)$.

Let us now examine a uniform convergence of a complex Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{-\mathrm{i}kx}. \tag{B.4.5}$$

Since x is real the magnitude

$$|e^{-\mathrm{i}kx}| = |\cos kx + \mathrm{i} \sin kx| = \sqrt{\cos^2 kx + \sin^2 kx} = 1,$$

and

$$|c_k e^{\mathrm{i}kx}| \leq |c_k| \quad \text{for all} \quad x.$$

Applying the Weierstrass test (Theorem B.20) we immediately deduce that

THEOREM B.22. *If the coefficients* $c_k = \frac{1}{2}(a_k \pm ib_k)$ *of a complex Fourier series* (B.4.5) *are such that*

$$\sum_{k=-\infty}^{\infty} |c_k| = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \sqrt{a_k^2 + b_k^2} < \infty \tag{B.4.6}$$

then the Fourier series converges uniformly to a continuous function $f(x)$. *Moreover, the coefficients* c_k *are equal to the Fourier coefficients of the function* $f(x)$.

Although Theorem B.22 gives conditions guaranteeing the convergence of a Fourier series, it does not tell if the limit $f(x)$ is the original function $\tilde{f}(x)$ for which the Fourier series was derived. Indeed, knowing that the coefficients are derived by integration one may suspect that the functions $f(x)$ and $\tilde{f}(x)$ differ at a finite, or even a countable, number of point. In fact, $\tilde{f}(x)$ may not be continuous. However, it can be shown that if a function is periodic and regular enough its Fourier series converges uniformly to the very function it got derived from.

THEOREM B.23. *Let $f(x)$ be 2π periodic and piecewise C^1. If $f(x)$ is continuous on an interval (a, b) then its Fourier series converges uniformly to $f(x)$ on any subinterval $[a+\delta, b-\delta]$ where $\delta > 0$.*

In other words, as long as we stay away from the discontinuities of $\tilde{f}(x)$ its Fourier series converges uniformly.

The uniform convergence of a Fourier series, as sanctioned by Theorem B.22, requires that the Fourier coefficients c_k approach 0 when $k \to \infty$. However, the convergence of the Fourier coefficients to zero is only a necessary condition. The coefficients need to converge fast enough to guarantee that the sum is finite. As we know from elementary calculus these coefficients must converge to zero faster then $\frac{1}{k}$. If, for example,

$$|c_k| \leq \frac{M}{|k|^\beta} \qquad \text{for all } k \text{ sufficiently large, where } \beta > 1, \tag{B.4.7}$$

then using the standard ratio test we can conclude that the series of coefficients c_k converges absolutely. This yields, according to the uniform convergence test (B.4.6), the uniform convergence of the Fourier series.

Note also that the faster the Fourier coefficients converge to zero the smoothes the limit function become. Indeed, suppose that for some positive integer n

$$|c_k| \leq \frac{M}{|k|^{\beta+n}} \qquad \text{for all } k \text{ sufficiently large, where } \beta > 1. \tag{B.4.8}$$

Then, we can differentiate the Fourier series up to n-times obtaining always, according to Proposition B.21, a uniformly convergent series.

THEOREM B.24. *If the Fourier coefficients are such that*

$$\sum_{k=-\infty}^{\infty} k^n |c_k| < \infty, \tag{B.4.9}$$

for some nonnegative integer n, then the Fourier series converges to a 2π periodic function $\tilde{f}(x)$ which is n times continuously differentiable. Moreover, for any $m \leq n$, the m times differentiated Fourier series converges uniformly to the corresponding derivative $\tilde{f}^{(m)}(x)$.

Hence, we may analyze the smoothness of a function by looking at how fast its Fourier coefficients approach zero. For example, if they converge to zero exponentially the function is infinitely differentiable but not necessarily analytic.

Convergence in the Norm. In order to be able to discuss this important aspect of convergence of Fourier series we need to take a short detour introducing the concept of a Hilbert space. The precise definition of a Hilbert space, which is rather technical, is beyond the scope of these notes. We will therefore present here a "working" version which will enable us to deal with Fourier series of functions such as, for example, the delta function.

Definition B.25. A complex-valued function $f(x)$ is said to be *square-integrable* on the $[-\pi, \pi]$ if

$$||f|| \equiv \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty. \tag{B.4.10}$$

In particular, but not only, every piecewise continuous function is square-integrable. Other, more singular, functions may also be square-integrable, e.g., x^{-r} is as long as $r < \frac{1}{2}$. Note also, that a function may not be square-integrable on some intervals while it is on some others.

The set of all square-integrable functions on $[-\pi, \pi]$, which is usually denoted as $\mathrm{L}^2[-\pi, \pi]$, is a complex vector space with the pointwise operations of addition and multiplication by a scalar. This is a strait forward consequence of the triangle inequality

$$|af(x) + bg(x)| \leq |a||f(x)| + |b||g(x)|.$$

Moreover, it is a normed vectors space with the L^2-norm given by (B.4.10) provided, we identify as one any two functions which differ at most on a *measure zero* set (for example, at a finite number of points of $[-\pi, \pi]$)[2]. L^2 space is a very special normed vector space as the L^2-norm is based on the (Hermitian) inner product

$$\langle f; g \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx. \tag{B.4.11}$$

[2] Such an identification is necessary as otherwise the L^2-norm would be zero for other than the constant zero function. In fact, it would not be a norm according to its definition A.1.8.

Indeed, $\langle f; f\rangle = ||f||^2$. The inner product is well defined and finite for any two square-integrable functions thanks to the Cauchy-Schwarz inequality.

EXAMPLE B.26. Consider a square-integrable function $f(x)$ on $[-\pi, \pi]$. Its Fourier coefficients are inner products of f with the exponential functions:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}dx = \langle f; e^{\mathrm{i}kx}\rangle.$$

Note also that as shown in (B.3.7)

$$||e^{\mathrm{i}kx}||^2 = \langle e^{\mathrm{i}kx}; e^{\mathrm{i}kx}\rangle = 1,$$

while

$$\langle e^{\mathrm{i}kx}; e^{\mathrm{i}lx}\rangle = 0 \qquad \text{if} \quad k \neq l.$$

We say that the infinite system of complex exponentials is an orthonormal system of functions (vectors) in $\mathrm{L}^2[-\pi, \pi]$, i.e., all elements have unit length and are mutually orthogonal as the scaler product of any two different elements vanish.

The sequence of square-integrable functions f_n converges in the L^2-norm to $g(x) \in \mathrm{L}^2[-\pi, \pi]$ if

$$||f_n(x) - g(x)||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(x) - g(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty. \tag{B.4.12}$$

Hence, let us consider the convergence of the sequence of partial sums $s_n(x)$ of the Fourier series of a square-integrable function $f(x)$. As each element of the sequence $s_n(x)$ is obviously square-integrable we can evaluate $||f - s_n||^2$. Using the definition of the sequence $s_n(x)$, and the linearity and symmetry of the inner product we obtain that

$$\begin{aligned} ||f - s_n||^2 &= ||f||^2 - 2\langle f; s_n\rangle + ||s_n||^2 = ||f||^2 - 2\sum_{k=-n}^{n} \overline{c_k}\langle f; e^{\mathrm{i}kx}\rangle + ||s_n||^2 \\ &= ||f||^2 - 2\sum_{k=-n}^{n} \overline{c_k}c_k + ||s_n||^2 = ||f||^2 - 2\sum_{k=-n}^{n} \overline{c_k}c_k + \sum_{k=-n}^{n} |c_k|^2 \\ &= ||f||^2 - ||s_n||^2. \end{aligned} \tag{B.4.13}$$

As the left hand side is always nonnegative

$$\sum_{k=-n}^{n} |c_k|^2 \leq ||f||^2$$

for any integer n. We have therefore proved the following:

PROPOSITION B.27 (Bessel's inequality). *If the function f is square-integrable on $[-\pi, \pi]$ then its Fourier coefficients are such that*

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right) = \sum_{-\infty}^{\infty} |c_k|^2 \leq ||f||^2. \tag{B.4.14}$$

This immediately implies that the individual Fourier coefficients (both complex and real) of a square-integrable function approach zero as $|k|$ approaches infinity. The convergence of the series (B.4.14) requires, in fact, more. The Fourier coefficients must tend to zero fast enough. If we postulate, as we did before in (B.4.7), that

$$|c_k| \leq \frac{M}{|k|^\beta} \qquad \text{for sufficiently large} \quad k, \tag{B.4.15}$$

then selecting $\beta > \frac{1}{2}$ guarantees the convergence already. Note this is a slower rate of decay than the one needed for the uniform convergence. One may have a Fourier series which converges in the L^2-norm but not uniformly.

If the Fourier series of a square-integrable function $f(x)$ converges in the L^2-norm to the very function it got derived from the Bessel's inequality becomes equality. Indeed, as evident from (B.4.13)

$$||f - s_n||^2 = ||f||^2 - ||s_n||^2 \tag{B.4.16}$$

for any Fourier partial sum $s_n(x)$. Therefore,

$$\lim_{n\to\infty} ||f - s_n|| = 0,$$

if and only if

$$||f||^2 = \lim_{n\to\infty} ||s_n||^2 = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{B.4.17}$$

This formula becomes a convenient criterion for the L^2-convergence of square-integrable functions.

PROPOSITION B.28 (Plancherel formula). *A Fourier series of a square integrable function $f \in L^2[-\pi,\pi]$ converges in the L^2-norm to $f(x)$ if and only if*

$$||f||^2 = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{B.4.18}$$

Before we finally show that the Fourier series of every L^2-function $f(x)$ converges in the L^2-norm to $f(x)$ let us note that one of the immediate consequences of the Plancheler formula is that a function is uniquely determined by its Fourier coefficients.

COROLLARY B.29. *Two functions $f, g \in L^2[-\pi,\pi]$ have the same Fourier coefficients if and only if $f = g$.*

The main result of the theory of Fourier series is the following convergence theorem.

THEOREM B.30. *Let $s_n(x)$ denote the n^{th} partial sum of the Fourier series of the square-integrable function $f(x) \in L^2[-\pi,\pi]$, then*

$$\lim_{n\to\infty} ||f - s_n|| = 0. \tag{B.4.19}$$

In other words, any square-integrable function on $[-\pi,\pi]$ can be uniquely represented by the infinite system[3] of complex exponentials $e^{\mathrm{i}kx}$, $k = 0, \pm1, \pm2, \ldots$.

PROOF OF THEOREM B.30. We shall provide here the prove for continuous functions only. Proving the validity of this statement for all square-integrable functions requires some extra work, see [**Carrier, Krook and Pearson**], and also [**Brown and Churchill**], and [**Kammler**]. According to Theorem B.23, if a function $f(x)$ is piecewise C^1 and continuous, its Fourier series converges to $f(x)$ everywhere, i.e.,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}kx}.$$

[3]This property of an infinite system of elements of a Hilbert space is known as the *completeness* of such a system. As the exponentials functions are orthonormal in $L^2[-\pi,\pi]$, this is the completeness of an orthonormal system of vectors.

Therefore, utilizing the fact that we are allowed to multiply and integrate uniformly convergent series term by term, we have

$$||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \int_{-\pi}^{\pi} f(x)\overline{f(x)} = \sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x)\overline{c}_k e^{-\mathrm{i}kx}dx$$
$$= \sum_{k=-\infty}^{\infty}\overline{c}_k c_k = \sum_{k=-\infty}^{\infty}|c_k|^2,$$

and Plancherel's identity (B.4.18) holds proving the convergence of the Fourier series. □

B.5. Fourier Transform

The *Fourier transform* may be viewed as a generalization of the Fourier series to an infinite interval. That is, take a Fourier series on a symmetric interval $[-l, l]$, and consider taking the limit as the length $l \to \infty$. The result of such a process is a Fourier series on an infinite interval. The corresponding representation of a function is given the name of the Fourier transform. Indeed, let us look for the representation of a $f(x)$ by a rescaled complex Fourier series (B.3.3) on an interval $[-l, l]$ in the following form

$$\sum_{r=-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{\hat{f}(k_r)}{l}e^{ik_r x} \tag{B.5.1}$$

where the sum is over the discrete set of frequencies

$$k_r = \frac{r\pi}{l}, \quad r = 0, \pm 1, \pm 2, \ldots, \tag{B.5.2}$$

to incorporate all trigonometric functions of period $2l$. Therefore, based on the form of Fourier coefficients (B.3.5) we get that

$$\hat{f}(k_r) = \frac{1}{\sqrt{2\pi}}\int_{-l}^{l} f(x)e^{-ik_r x}dx. \tag{B.5.3}$$

This allows us to pass to the limit $l \to \infty$, and to get the infinite integral

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{B.5.4}$$

known as the *Fourier transform* of the function $f(x)$. If the function $f(x)$ is piecewise continuous and decays to 0 reasonably fast as $x \to \pm\infty$, the Fourier transform $\hat{f}(k)$ is defined for all frequencies $-\infty < k < \infty$.

REMARK B.31. Note that the discrete frequencies (B.5.2) used to represent the function $f(x)$ on the interval $[-l, l]$ are equally spaced as

$$\triangle k = k_{r+1} - k_r = \frac{\pi}{l}.$$

As $l \to \infty$, the spacing between frequencies $\triangle k \to 0$, and the frequencies become more and more densely distributed. That is why in the Fourier transform limit we anticipate that all frequencies participate in representing a function.

Taking into account the fact that the discrete frequencies k_r are equally spaced allows us to re-write the Fourier series (B.5.1) as

$$\frac{1}{\sqrt{2\pi}} \sum_{r=-\infty}^{\infty} \hat{f}(k_r) e^{ik_r x} \triangle k. \tag{B.5.5}$$

This has the form of a Riemann sum for the function $\hat{f}(k)e^{ikx}$. Assuming that it converges as $\triangle k \to 0$ we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk. \tag{B.5.6}$$

The series (B.5.5) becomes a *Fourier integral* which reconstructs the function $f(x)$ as a continuous superposition of complex exponential functions e^{ikx}. For example, if the function $f(x)$ is piecewise continuously differentiable everywhere and it decays reasonably fast as $|x| \to \infty$, then the inverse Fourier integral (B.5.6) converges to $f(x)$ at all points of continuity. At the jump discontinuities it converges to the midpoint $\frac{1}{2}[f_+(x) - f_-(x)]$. Indeed, it can be shown[4] that

THEOREM B.32. *If the function $f(x)$ is piecewise continuous and square-integrable (for $-\infty < x < \infty$), then its Fourier transform $\widehat{f}(k)$ is well defined and square-integrable. Moreover, if the right and left hand limits $f_-(x)$, $f_+(x)$ and $f'_-(x)$, $f'_+(x)$ exist, then the Fourier integral (B.5.6) converges to the average*

[4] A rigorous proof of a more general statement can be found in [**Carrier, Krook and Pearson**].

value $\frac{1}{2}[f_-(x) + f_+(x)]$. *In particular, if* $f(x)$ *is continuously differentiable at a point* x*, then the Fourier integral equals the value of* $f(x)$.

Example B.33. Let us consider an exponentially decaying pulse

$$f(x) = e^{-a|x|}. \tag{B.5.7}$$

Its Fourier transform

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a-ik)x}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+ik)x}dx \\ &= \frac{1}{\sqrt{2\pi}}\left.\frac{e^{(a-ik)x}}{a-ik}\right|_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-(a+ik)x}}{a+ik}\right|_{0}^{\infty} = \sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}.\end{aligned} \tag{B.5.8}$$

Notice that the Fourier transform of our pulse $f(x)$ which is real and even, is itself real and even. The inverse Fourier transform gives

$$e^{-a|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ae^{ikx}}{k^2+a^2}dk = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a\cos kx}{k^2+a^2}dk. \tag{B.5.9}$$

Here the imaginary part of the integral vanishes as the integrant is odd.

We have been discussing the Fourier transform of the pulse function not without reason. First, it is interesting to notice that as $a \to 0$ the pulse approaches the constant function $g(x) \equiv 1$. Moreover, the limit of its Fourier transform (B.5.8) is

$$\lim_{a\to 0}\sqrt{\frac{2}{\pi}}\frac{2a}{k^2+a^2} = \begin{cases} 0, & k \neq 0 \\ \infty, & k = 0 \end{cases}. \tag{B.5.10}$$

Comparing this with the original construction (1.2.8) of the delta function as the limit of approximating functions we notice that

$$\delta(x) = \lim_{n\to\infty}\frac{n}{\pi(1+n^2x^2)} = \lim_{a\to 0}\frac{a}{\pi(a^2+x^2)}, \tag{B.5.11}$$

provided $n = 1/a$. This, in turn, allows us to write the Fourier transform of the constant function $f(x) \equiv 1$ as

$$\hat{f}(k) = \sqrt{2\pi}\delta(k). \tag{B.5.12}$$

Equivalently, this should mean that

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx. \tag{B.5.13}$$

The infinite integral on the right hand side does not converge[5]. However, from the definition of the delta function we have

$$\int_{-\infty}^{\infty} \delta(k)e^{-ikx}dk = e^{ik0} = 1, \tag{B.5.14}$$

which implies that the Fourier transform of the delta function is a constant function

$$\hat{\delta}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{-ikx}dx = \frac{e^{-ik0}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}. \tag{B.5.15}$$

To determine the Fourier transform of the delta function concentrated at an arbitrary position y, namely $\delta_y(x)$ we cite the following theorem:

THEOREM B.34.

- *If the Fourier transform of the function $f(x)$ is $\hat{f}(k)$, then the transform of $\hat{f}(x)$ is $f(-k)$.*
- *If the function $f(x)$ has the Fourier transform $\hat{f}(k)$, then the Fourier transform of the shifted function $f(x-y)$ is $e^{-iky}\hat{f}(k)$. By analogy, the transform of the product $e^{inx}f(x)$ is the shifted Fourier transform $\hat{f}(k-n)$.*

Therefore, according to (B.5.15) the Fourier transform of the delta function at $y \neq 0$ is

$$\hat{\delta}_y(k) = \frac{e^{-iky}}{\sqrt{2\pi}}. \tag{B.5.16}$$

Thus, combining the definitions of the Fourier transform and its inverse, as well as using the basic properties of the delta function leads to the identity

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-ik(x-y)}dxdk \tag{B.5.17}$$

valid for suitable functions.

[5]It may, however, be interpreted in the context of generalized functions.

Differentiation and Integration.

As our main objective in dealing with Fourier transforms is to show how they can be used to solve differential equations we embark now on the analysis of the process of differentiation and integration of these functions.

Let us consider the Fourier transform of the derivative of $f(x)$

$$\widehat{f'}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx, \tag{B.5.18}$$

assuming that $f'(x)$ is behaving well enough so the integral exists. Integrating by parts, and taking into account that $f(x)$ approaches 0 at infinities rapidly enough, we obtain that

$$\widehat{f'}(k) = \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{B.5.19}$$

Thus, we have that

Proposition B.35. *The Fourier transform of the derivative $f'(x)$ of a function is obtained by multiplying its Fourier transform by ik:*

$$\widehat{f'}(k) = ik\hat{f}(k). \tag{B.5.20}$$

Iterating the formula (B.5.20) yields that

Corollary B.36. *The Fourier transform of $f^{(n)}(x)$ is $(ik)^n \hat{f}(k)$.*

Consider now

$$g(x) = \int_{-\infty}^{x} f(y) dy.$$

We are interested in finding the Fourier transform $\hat{g}(k)$. To this end notice first that

$$\lim_{x \to -\infty} g(x) = 0, \quad \text{while} \quad \lim_{x \to +\infty} = \int_{-\infty}^{\infty} f(y) dy = c.$$

Therefore, consider the function $h(x) = g(x) - c\sigma(x)$ which decays to 0 at both $\pm\infty$. As the Fourier transform of the step function $\sigma(x)$ is

$$\hat{\sigma}(k) = \sqrt{\frac{2}{\pi}} \delta(k) - \frac{i}{k\sqrt{2\pi}} \tag{B.5.21}$$

we obtain that

$$\hat{h}(k) = \hat{g}(k) - c\sqrt{\frac{2}{\pi}} \delta(k) + \frac{ic}{k\sqrt{2\pi}}. \tag{B.5.22}$$

On the other hand,

$$h'(x) = f(x) - c\delta(x).$$

Applying Proposition B.35 we conclude after some manipulations that

$$\hat{g}(k) = \frac{\hat{f}(k)}{ik} + c\sqrt{\frac{\pi}{2}}\delta(k). \tag{B.5.23}$$

EXAMPLE B.37. Consider the boundary value problem

$$-u'' + \omega^2 u = h(x), \quad -\infty < x < \infty, \tag{B.5.24}$$

where $\omega > 0$, and the solution is assumed square-integrable implying that u approaches 0 at $\pm\infty$. Taking the Fourier transform of both sides of the differential equation renders an algebraic equation relating the Fourier transforms of u and h. Indeed,

$$k^2\widehat{u}(k) + \omega^2\widehat{u} = \widehat{h}(k). \tag{B.5.25}$$

The transformed equation can now be solved for the Fourier transform of the solution

$$\widehat{u}(k) = \frac{\widehat{h}(k)}{k^2 + \omega^2}. \tag{B.5.26}$$

Using the Fourier formula (B.5.6) we reconstruct the solution as

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\widehat{h}(k)e^{ikx}}{k^2 + \omega^2} dk. \tag{B.5.27}$$

One of the most important and interesting cases is the one in which $h(x) = \delta_y(x)$, that is when the forcing term is in the form of the impulse concentrated at y. The corresponding solution is the Green's function $u(x) = G(x, y)$. According to (B.5.26) and Theorem B.34, its Fourier transform with respect to x is

$$\widehat{G}(k, y) = \frac{e^{-iky}}{k^2 + \omega^2}. \tag{B.5.28}$$

Notice first that according to Example B.33 the inverse Fourier transform of the reciprocal of $k^2 + \omega^2$ is

$$\frac{e^{-\omega x}}{2\omega}.$$

Secondly, the exponential term in the numerator implies a shift. Therefore, the Green's function for our boundary-value problem is

$$G(x,y) = \frac{e^{-\omega|x-y|}}{2\omega}.$$

The Green's function satisfies the differential equation everywhere except at $x = y$ where it has a jump discontinuity of unit magnitude. It also satisfies the boundary conditions as it decays to 0 as $|x| \to \infty$. Invoking the general superposition principle for Green's function we obtain the solution to our boundary-value problem with an arbitrary forcing term as

$$u(x) = \int_{-\infty}^{\infty} G(x,y)h(y)dy = \frac{1}{2\omega}\int_{-\infty}^{\infty} e^{-\omega|x-y|}h(y)dy. \tag{B.5.29}$$

Note that the Green's function $G(x,y)$ depends only on the difference $x - y$ and that the solution $u(x)$ takes the form of the *convolution*. Namely,

$$u(x) = \int_{-\infty}^{\infty} g(x-y)h(y)dy = g(x) * h(x). \tag{B.5.30}$$

On the other hand, as we saw earlier in (B.5.26), the Fourier transform of the solution to our boundary value problem is a product of the Fourier transforms of the Green's function and the forcing term. Indeed, we have that:

THEOREM B.38. *The Fourier transform of the convolution $u(x) = g(x) * h(x)$ of two functions is, up to multiple, the product of their Fourier transforms*

$$\widehat{u}(k) = \sqrt{2\pi}\,\widehat{g}(k)\widehat{h}(k). \tag{B.5.31}$$

By symmetry, the Fourier transform of the product $h(x) = f(x)g(x)$ is a multiple of the convolution of their Fourier transforms

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}}\widehat{f}(k) * \widehat{g}(k). \tag{B.5.32}$$