ON THE SPECTRA OF CERTAIN DIRECTED PATHS

C.M. DA FONSECA AND J.J.P. VEERMAN

Abstract. We describe the eigenpairs of a special kind of tridiagonal matrices related with problems on traffic on a one lane road. Some numerical examples are provided.

1. Introduction

The n-by-n tridiagonal matrix

$$Q_\rho = \begin{pmatrix} 0 & \rho & & & \\ 1-\rho & \ddots & \ddots & & \\ & \ddots & \ddots & \rho & \\ & & 1-\rho & 0 & \rho \\ & & & 1 & 0 \end{pmatrix},$$

where ρ is an arbitrary real number in $(0,1)$, is of fundamental importance in understanding the dynamics of Newtonian particles in a chain with (generally) asymmetric nearest neighbor interactions, presuming n to be large.

For an eigenvector v of Q_ρ associated to the eigenvalue r, i.e., for an eigenpair $\{r, v\}$, $\{1-r, v\}$ is an eigenpair of $I - Q_\rho$. The matrix Q_ρ derives its importance from the fact that $I - Q$ is the directed graph Laplacian (cf. e.g. [1, 5, 6, 7]) associated to an important system of linear differential equations, modeling a simple instance of flocking behavior related to studies of automated traffic on a single lane road. In this note we provide some expressions for the eigenvalues and eigenvectors of the matrix Q_ρ. For that purpose, for a positive real number κ, we first analyze the location of the zeroes of the polynomial

$$f(x) \stackrel{\text{def}}{=} g(x) - g(h(x)) \,, \tag{1.1}$$

where

$$g(x) \stackrel{\text{def}}{=} x^{n+1} - x^{n-1} \quad \text{and} \quad h(x) \stackrel{\text{def}}{=} \frac{\kappa}{x} \,.$$

The method presented here relies on the observation that the eigenvalue equation for Q_ρ can be rewritten as a two dimensional recursive system with appropriate boundary condition near 0 and n indexes. This procedure can be found for instance in [4]. This case can also be seen in the perspective of the orthogonal polynomials theory as in [3].

2. The zeroes of a polynomial

The main tool in analyzing the eigenvalues of the matrix Q_ρ is the analysis of the location of the zeroes of the polynomial $f(x)$ defined in (1.1). This polynomial is also of independent interest (see [8]). Henceforth, the square root stands for the root in the upper half plane minus the negative real axis.

Date: May 2008.

2000 *Mathematics Subject Classification.* 15A18; 15A09; 15A47.

Key words and phrases. Tridiagonal matrix, directed paths, eigenvalues, eigenvectors, location of eigenvalues.

This work was supported by CMUC - Centro de Matemática da Universidade de Coimbra.

In the statement of Theorem 2.1 we use the following equation, where $\kappa > 0$ and ϕ are real variables:

$$\frac{(1-\kappa)}{(1+\kappa)}\cot\phi = \cot n\phi\ . \tag{2.1}$$

For example, if $\kappa = 1$, this is equivalent to $\cos n\phi\ = 0$, and its solutions are given by $\phi_\ell = \pm\frac{(2\ell+1)\pi}{2n}$, for $\ell = 0, \dots, n-1$.

Theorem 2.1. *For any positive real number κ, the equation (1.1) has $2n+2$ roots. Two of these are the fixed points of h given by $\pm\sqrt{\kappa}$. The remaining $2n$ roots have period 2 under the involution h and are given as follows:*

i) *If $\kappa \geq 1$: n roots are given by $\sqrt{\kappa}\,e^{i\phi_\ell}$, where $\phi_\ell \in \left(\frac{\ell\pi}{n}, \frac{(\ell+1)\pi}{n}\right)$, for $\ell \in \{0, \dots, n-1\}$, solves (2.1); the remaining roots are the images under h of these or: $\sqrt{\kappa}\,e^{-i\phi_\ell}$, respectively.*

ii) *If $\kappa \in [\frac{n-1}{n+1}, 1)$: Identical to i).*

iii) *If $\kappa \in (0, \frac{n-1}{n+1})$: $n-2$ roots are given by $\sqrt{\kappa}\,e^{i\phi_\ell}$, where $\phi_\ell \in \left(\frac{\ell\pi}{n}, \frac{(\ell+1)\pi}{n}\right)$, for $\ell \in \{1, \dots, n-2\}$, solves (2.1); $n-2$ are images of these under h; the remaining roots are $x_0 \in (\sqrt{\kappa}, 1)$ and its images under h and multiplication by -1. We have $x_0 = 1 - \frac{1}{2}(1-\kappa^2)\kappa^{n-1} + \mathcal{O}(\kappa^{2n-2})$.*

Note that when $\kappa = \frac{n-1}{n+1}$, the fixed points of h coincide with other roots and thus having higher multiplicity (namely 2). When multiple roots are present, we count them with (algebraic) multiplicity.

Proof. We have $x^{n+1}f(x) = (x^{2n+2} - \kappa^{n+1}) - x^2(x^{2n-2} - \kappa^{n-1})$. This polynomial has exactly $2n+2$ non-zero roots and these are also the roots of the equation $f(x) = 0$ (always counting multiplicity). Two roots are given by the only fixed points of h, namely $\pm\sqrt{\kappa}$. Our strategy here is to then to find n roots of $f(x)$ in the upper half plane. Since h is an involution the remaining n roots are then found by taking their image under h to get the roots in the lower half plane.

If we substitute $x = \sqrt{\kappa}\,e^{i\phi}$ into the equation $f(x) = 0$ we get $\kappa\sin{(n+1)\phi} - \sin(n-1)\phi = 0$, which is equivalent to: $\kappa\,(\sin n\phi\cos\phi + \cos n\phi\sin\phi) = \sin n\phi\cos\phi - \cos n\phi\sin\phi$. Collecting similar terms then gives

$$(1-\kappa)\sin n\phi\cos\phi = (1+\kappa)\cos n\phi\sin\phi\quad . \tag{2.2}$$

This in turn gives Equation (2.1) upon division by $(1+\kappa)\sin n\phi\sin\phi$.

To prove i), first note that the case $\kappa = 1$ follows directly from Equation (2.1). In the remaining cases the coefficient $\frac{1-\kappa}{1+\kappa}$ is negative. A straightforward graphical inspection of equation (2.1) (see Figure 4.2, first figure) establishes the existence of n solutions ϕ_ℓ, one in each interval $\left(\frac{\ell\pi}{n}, \frac{(\ell+1)\pi}{n}\right)$, for $\ell = 0, 1, \dots, n-1$.

Now we prove ii). In this case the coefficient in Equation (2.1) is greater than zero. We see upon inspecting the graphical solution (Figure 4.2, second figure), that in the interval $(0, \pi)$, the equation (2.1) has $n-2$ natural solutions, one in each interval $\left(\frac{\ell\pi}{n}, \frac{(\ell+1)\pi}{n}\right)$, for $\ell = 1, \dots, n-2$. To see whether there are roots in the remaining two intervals for $\ell = 0$ and $\ell = n-1$, divide Equation (2.2) by $(1-\kappa)\cos n\phi\cos\phi$. We then get

$$\frac{(1+\kappa)}{(1-\kappa)}\ \tan\phi = \tan n\phi\ .$$

This equation has a solution (not equal to 0 or π) in each of the two intervals if

$$\frac{\partial}{\partial\phi}\Big|_{\phi=0}\frac{1+\kappa}{1-\kappa}\ \tan\phi > \frac{\partial}{\partial\phi}\Big|_{\phi=0}\tan n\phi\ ,$$

which is equivalent to

$$\kappa > \frac{n-1}{n+1}\ .$$

Since the roots of a polynomial are continuous functions of the coefficients, we get roots of multiplicity 2 at $\pm\sqrt{\kappa}$, when $\kappa = \frac{n-1}{n+1}$.

The proof of iii) runs parallel to the previous except that now there are no solutions (other than 0 and π) in the intervals labeled $\ell = 0$ and $\ell = n-1$. These solutions plus their images under h give us $2n-2$ roots of f. Straightforward arguments ($f(\sqrt{\kappa}) = 0$, $f(1) > 0$, and $f'(\sqrt{\kappa}) < 0$) lead to the insight that there is a new real positive root in $(\sqrt{\kappa}, 1)$. Its image under h then yields a root in $(\kappa, \sqrt{\kappa})$. Since $x^{n+1}f(x)$ is even, we can multiply these roots by -1 to get two more.

Applying Newton's Method to the starting point 1, we get for one of the roots (up to $\mathcal{O}(\kappa^{2n-2})$):

$$\bar{x} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(1-\kappa^2)\kappa^{n-1}}{2(1+\kappa^{n-1})} \approx 1 - \frac{1}{2}(1-\kappa^2)\kappa^{n-1} .$$

The precise estimate follows from the fact that Newton's Method converges quadratically. The other roots are obtained by taking the images under h and multiplication by -1. □

3. The eigenpairs of Q_ρ

In this section, we establish formulas for the eigenpairs of Q_ρ. It turns out that Theorem 2.1 can be translated rather easily to give our result here.

In the statement of Theorem 3.1 we use the following equation, where $\rho \in (0,1)$ and ϕ are real variables:

$$(2\rho - 1)\cot\phi = \cot n\phi . \tag{3.1}$$

Theorem 3.1. *For any real number $\rho \in (0,1)$, the matrix Q_ρ has n eigenvalues (counting multiplicity). They are given as follows:*

i) If $\rho \in (0, \frac{1}{2}]$**:** *The n eigenvalues are given by $2\sqrt{\rho(1-\rho)}\cos\phi_\ell$, where $\phi_\ell \in \left(\frac{\ell\pi}{n}, \frac{(\ell+1)\pi}{n}\right)$, for $\ell \in \{0, \dots, n-1\}$, solves (3.1).*

ii) If $\rho \in (\frac{1}{2}, \frac{n+1}{2n}]$**:** *Identical to i).*

iii) If $\rho \in (\frac{n+1}{2n}, 1)$**:** *$n-2$ eigenvalues are given by $2\sqrt{\rho(1-\rho)}\cos\phi_\ell$, where $\phi_\ell \in \left(\frac{\ell\pi}{n}, \frac{(\ell+1)\pi}{n}\right)$, for $\ell \in \{1, \dots, n-2\}$, solves (3.1); the remaining two are given by $\pm\left(1 - \frac{(2\rho-1)^2}{2\rho^2}\left(\frac{1-\rho}{\rho}\right)^{n-1}\right)$, with an error $\mathcal{O}\left(\left(\frac{1-\rho}{\rho}\right)^{2n-2}\right)$ as n tends to infinity.*

It is well known (cf., e.g., [2, p.28]) that if Q_ρ is irreducible, then the eigenvalues are all distinct, and in the case Q_ρ is sign-symmetric, they are all real.

Proof. Let $v = (v_1, \cdots, v_n)^T \in \mathbb{C}^n$ be an eigenvector of Q_ρ associated to the eigenvalue $r \in \mathbb{C}$. The idea is to replace the equation $Q_\rho v = rv$ by a *local* version modified by adequate boundary conditions. This (equivalent) local reformulation of the eigenpair equation for Q_ρ is:

$$\text{For } j = 1, \dots, n \ : \ \begin{pmatrix} v_j \\ v_{j+1} \end{pmatrix} = C^j \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \quad \text{and} \quad \begin{cases} v_0 = 0 \\ v_{n+1} = v_{n-1} \end{cases} . \tag{3.2}$$

Here the matrix C is defined by

$$C = \begin{pmatrix} 0 & 1 \\ -\frac{1-\rho}{\rho} & \frac{r}{\rho} \end{pmatrix} ,$$

and we will refer to the last two conditions as boundary condition 1 and 2, respectively. The aim is then to find n pairs $\{r, v\}$ satisfying (3.2).

Let $h : \mathbb{C} \to \mathbb{C}$ be the involution given by: $h(x) = \frac{1-\rho}{\rho x}$. The eigenvalues $x_\pm$ of C satisfy

$$x_\pm = h(x_\mp) \quad \text{and} \quad r = \rho \operatorname{tr} C = \rho(x_+ + h(x_+)) . \tag{3.3}$$

Since the assumption of $x_+ \neq h(x_+)$ yields a procedure that produces all n distinct pairs $\{r, v\}$ satisfying (3.2), we can consequently omit the case where $x_+ = h(x_+)$. Hence, let us assume that $x_+ \neq h(x_+)$. In this case C is diagonalizable, with eigenvectors $(1, x_\pm)^T$. Denote by x either of the two eigenvalues. Any solution of the recursion (3.2) can be written as

$$v_j = c_+ x^j + c_- h(x)^j \ .$$

Boundary condition 1 implies that $c_- = -c_+$ (and v_j is non-zero if $x \neq h(x)$). We can take $c_+ = 1$ without loss of generality. Boundary condition 2 becomes

$$x^{n+1} - x^{n-1} - \big(h(x)^{n+1} - h(x)^{n-1}\big) = 0 \ . \tag{3.4}$$

After setting $\kappa = \frac{1-\rho}{\rho}$ this is equivalent to Equation (1.1).

Finding the spectrum of Q_ρ is equivalent to get n values for $\rho(x+h(x))$ (counting multiplicity), where x is determined as (3.4). Hence, from Theorem 2.1, we may establish the result. □

About the eigenvectors of Q_ρ, we may conclude the following proposition.

Corollary 3.2. *The eigenvectors of Q_ρ are given by $v_j = x^j - h(x)^j$, where x satisfies (3.4).*

For the case of $\rho = 1/2$, we conclude that if $v = (v_1, \ldots, v_n)$ is an eigenvector associated to the eigenvalue $\cos \frac{(2\ell+1)\pi}{2n}$, with $\ell \in \{0, \ldots, n-1\}$, then

$$v_j = \sin \frac{(2\ell - 1)j}{2n}\, \pi$$

for $j = 1, \ldots, n$.

4. Numerical Examples

To end this note, we present below two graphs of the set of eigenvalues of Q_ρ that were evaluated using MAPLE, for $n = 5$ and $n = 6$, respectively, and for ρ in $(0, 1)$. We also present a sketch of the solution of Equation (2.1) in two cases.

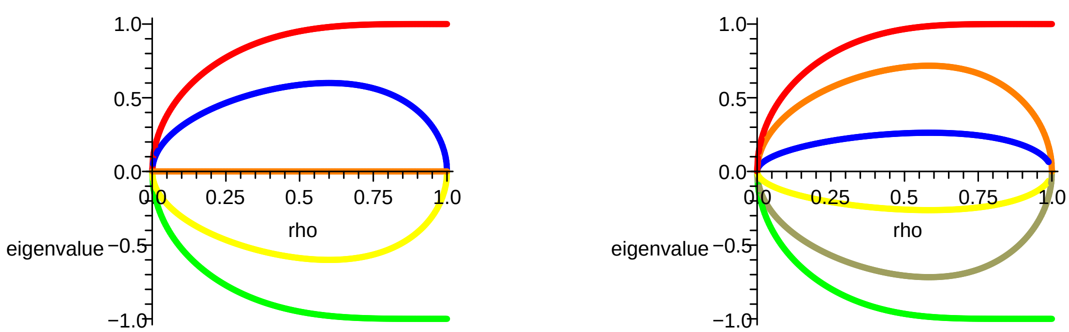

Figure 4.1. *The eigenvalues of Q_ρ as function of ρ for $n = 5$ and $n = 6$.*

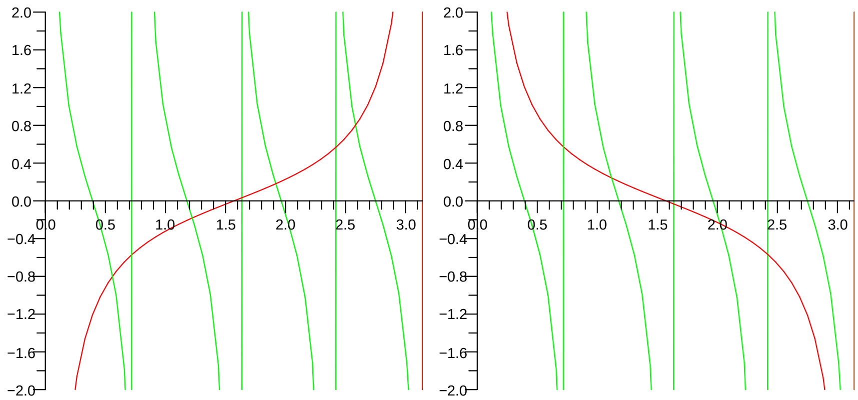

FIGURE 4.2. *When* $\kappa > 1$ *(first figure) Equation (2.1) has* $2n$ *solutions in* $(-\pi, \pi)$*. When* $0 < \kappa < 1$*, there are only* $2n - 2$ *(see second figure). Here the only solutions in* $(0, \pi)$ *are shown. In this figure* $n = 4$*.*

References

[1] J.S. Caughman, J.J.P. Veerman, Kernels of directed graph Laplacians, *Electron. J. Combin.* **13** (1) Research Paper 39 (2006).

[2] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[3] C.M. da Fonseca, On the location of the eigenvalues of Jacobi matrices, *Appl. Math. Lett.* **19** 1168-1174 (2006) .

[4] G.L. Kotkin, V.G. Serbo, *Problemas de Mecánica Clásica*, Mir, Moscow, 1980.

[5] J.J.P. Veerman, J.S. Caughman, G. Lafferriere, A. Williams, Flocks and formations, *J. Stat. Phys.* **121** (5-6) 901-936 (2005).

[6] J.J.P. Veerman, B.D. Stošić, A. Olvera, Spatial instabilities and size limitations of flocks, *Netw. Heterog. Media* **2** (4) (2007) 647-660.

[7] J.J.P. Veerman, B.D. Stošić, F. M. Tangerman, Automated traffic and the finite size resonance, *J. Stat. Phys.*, submitted.

[8] J.J.P. Veerman, On Flock Stability, In Preparation.

[9] A. Williams, G. Lafferriere, J.J.P. Veerman, Stable motions of vehicle formations, Proc. 44th Conference of Decision and Control, 72-77, December 2005.

CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: cmf@mat.uc.pt

Department of Mathematics & Statistics, Portland State University, Portland, OR 97201, USA
Center for Physics and Biology, Rockefeller University, New York, NY 10021, USA
E-mail address: veerman@pdx.edu